

# Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity

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## Abstract

We use the perturbation theory to build solitary wave solutions  $\phi_\omega(x)e^{-i\omega t}$  to the nonlinear Dirac equation in  $\mathbb{R}^n$ ,  $n \geq 1$ , with the Soler-type nonlinear term  $f(\psi^* \beta \psi) \beta \psi$ , with  $f(\tau) = |\tau|^k + o(|\tau|^k)$ ,  $k > 0$ , which is continuous but not necessarily differentiable. We obtain the asymptotics of solitary waves in the nonrelativistic limit  $\omega \lesssim m$ ; these asymptotics are important for the linear stability analysis of solitary wave solutions. We also show that in the case when the power of the nonlinearity is Schrödinger charge-critical ( $k = 2/n$ ), then one has  $Q'(\omega) < 0$  for  $\omega \lesssim m$ , with  $Q(\omega)$  being the charge of the corresponding solitary wave; this implies the absence of the degeneracy of zero eigenvalue of the linearization at this solitary wave.

## 1 Introduction

Construction of solitary wave solutions in Dirac-type systems has a long history. In the three-dimensional nonlinear Dirac equation, the solitary waves were numerically constructed by Soler [Sol70] and then proved to exist in [Vaz77, CV86, Mer88, ES95]. In the Dirac–Maxwell system, solitary waves were obtained numerically [Gro66, Wak66, Lis95] and then analytically [EGS96] (for  $\omega \in (-m, 0)$ ) and [Abe98] (for  $\omega \in (-m, m)$ ); for an overview of these results, see [ES02]. A perturbation method for the construction of solitary waves in the nonlinear Dirac equation was used in [Oun00]. This work was later followed in [Gua08, CGG14] and also generalized to the Einstein–Dirac and Einstein–Dirac–Maxwell systems [RN10a, Stu10, RN10b] and to the Dirac–Maxwell system [CS12]. Our aim here is to make the perturbative approach of the seminal work [Oun00] rigorous for the important case of lower order nonlinearities. The usefulness of such an approach is that it gives the asymptotic behaviour of solitary waves which is needed for the study of their stability properties. The bifurcation approach (in the nonrelativistic limit  $\omega \gtrsim -m$ ) to obtain Dirac–Maxwell solitary waves as perturbations of solitary waves of the Choquard equation was developed in [CS12].

In the present analysis, we use the bifurcation approach to construct solitary wave solutions to the nonlinear Dirac equation with scalar-type self-interaction, known as the Soler model [Iva38, Sol70]:

$$i\partial_t \psi = D_m \psi - f(\bar{\psi} \psi) \beta \psi, \quad \psi(x, t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \quad n \geq 1, \quad (1.1)$$

where  $D_m = -i\alpha \cdot \nabla + \beta m$  is the free Dirac operator, with  $\alpha = (\alpha^j)_{1 \leq j \leq n}$ ,  $\alpha^j$  and  $\beta$  being the self-adjoint  $N \times N$  Dirac matrices (see Remark 1.1 below for a possible choice of such matrices);  $m > 0$  is the mass. We use the standard Physics notation  $\bar{\psi} := \psi^* \beta$ . The real-valued function

$$f \in C(\mathbb{R}), \quad f(\tau) = |\tau|^k + o(|\tau|^k), \quad \tau \in \mathbb{R}, \quad k > 0, \quad (1.2)$$

describes the nonlinearity. We obtain solitary wave solutions to (1.1) in the nonrelativistic limit,

$$\phi_\omega(x)e^{-i\omega t}, \quad \phi_\omega \in H^1(\mathbb{R}^n), \quad \omega \lesssim m,$$

building them as bifurcations from solitary waves of nonlinear Schrödinger equation; the construction provides description of solitary waves which we will need for the analysis of their spectral stability (presence or absence of eigenvalues with positive real part in the spectrum of the linearization at a solitary wave), continuing the program started in [BC16]. We refer to that work for more details and the background on the subject.

Most common models considered by physicists and chemists (e.g. [Rañ83]) are pure powers  $f(\tau) = |\tau|^k$ , usually cubic ( $k = 1$ ) and quintic ( $k = 2$ ). As we already mentioned, there have been several implementations of constructing solitary waves via the bifurcation method for such models, but these approaches did not allow one to handle the low regularity case, such as  $f(\tau) = |\tau|^k$ , with  $k \in (0, 1)$ , when  $f(\tau)$  is no longer differentiable at  $\tau = 0$ , so that the derivative of  $f$  would contribute a singularity if the Lorentz scalar  $\bar{\phi}_\omega \phi_\omega := \phi_\omega^* \beta \phi_\omega$  vanished. On the other hand, this low regularity case also corresponds to the interesting “Schrödinger charge-subcritical” case, when  $k \in (0, 2/n)$  (with  $n \geq 2$ ), so that the “groundstate” solitary waves for NLS are stable (groundstate is understood in the sense of [BL83a]: it is a strictly positive, spherically symmetric, decaying solution to the stationary NLS). With these values of  $k$ , one can compare stability properties in both models, pushing further the discussion from [CMKS<sup>+</sup>16]. We overcome the difficulties resulting from the low regularity of  $f$  in the nonrelativistic limit  $\omega \lesssim m$ , constructing solitary waves for arbitrary  $f$  from (1.2). The main points are to base the construction on the Schauder fixed point theorem (instead of the contraction mapping principle which is not available to us when  $f(\tau)$  is not Lipschitz) and to prove that  $\phi_\omega(x)^* \beta \phi_\omega(x)$  is bounded from below by  $c\phi_\omega(x)^* \phi_\omega(x)$  with some  $c \in (0, 1)$ , for  $\omega$  sufficiently close to  $m$ . In the case when  $f$  is differentiable away from the origin, we will additionally prove uniqueness of  $\phi_\omega$  (up to the symmetry transformations) and also its differentiability with respect to  $\omega$ .

We note that quintic nonlinear Schrödinger equation in (1+1)D and the cubic one in (2+1)D are “charge critical”, in the sense that the equation has the same scaling as the charge and as a consequence all groundstate solitary waves have the same charge. As a consequence, by [VK73], the linearization at any solitary wave has a  $4 \times 4$  Jordan block at  $\lambda = 0$ . We mention that there is also a blow-up phenomenon in the charge-critical as well as in the charge-supercritical cases; see in particular [ZSS71, ZS75, Gla77, Wei83, Mer90]. In the present work, we will show that, on the contrary, for the nonlinear Dirac with the “Schrödinger charge-critical” power  $f(\tau) = |\tau|^k$ , with  $k = 2/n$  (in any dimension  $n \geq 1$ ) the charge of solitary waves is no longer the same, satisfying  $\partial_\omega Q(\phi_\omega) < 0$  for  $\omega \lesssim m$ , where  $Q(\phi_\omega) = \int_{\mathbb{R}^n} \phi_\omega(x)^* \phi_\omega(x) dx$  is the corresponding charge. This reduces the degeneracy of the zero eigenvalue of the linearization at the corresponding solitary wave (see e.g. [BCS15]). In formal agreement with the Vakhitov–Kolokolov stability criterion [VK73], one expects that the solitary wave solutions to the nonlinear Dirac equation in the nonrelativistic limit  $\omega \lesssim m$  are spectrally stable; indeed, this has been verified numerically in one- and in two-dimensional cases [CPS16, CMKS<sup>+</sup>16].

Let us make a few more remarks on the relation to the Vakhitov–Kolokolov stability criterion [VK73]. In the case of self-interacting classical spinor fields, although the relation of the sign of the quantity  $\partial_\omega Q(\phi_\omega)$  entering the Vakhitov–Kolokolov stability criterion and the presence or absence of positive eigenvalues in the spectrum is no longer clear, the vanishing of  $\partial_\omega Q(\phi_\omega)$ , together with the energy vanishing, indicate the collision of point eigenvalues at the origin; for more details, see [BCS15]. Moreover, we point out that, unlike in the Schrödinger equation, in the Dirac context eigenvalues with nonzero real parts can emerge not only from the collision of purely imaginary eigenvalues at the origin, but also from collision of purely imaginary eigenvalues away from the origin [CMKS<sup>+</sup>16] and directly from the essential spectrum [BPZ98]. Thus the Vakhitov–Kolokolov criterion is insufficient for the characterization of the spectral stability.

Here is the plan of the present analysis. The main results stated in Section 2 are the existence of solitary waves for the case of a continuous nonlinearity (Theorem 2.1) and the improvement for the case of nonlinearity differentiable everywhere except perhaps at zero (Theorem 2.2). Theorem 2.1 is proved in Sections 3 (the Schauder fixed point theorem), Section 4 (positivity of  $\bar{\phi}_\omega \phi_\omega := \phi_\omega^* \beta \phi_\omega$ ), and Section 5 (accurate es-

timates on the error terms). Theorem 2.2 is proved in Sections 6 (regularity of mapping  $\omega \mapsto \phi_\omega$ ) and 7 (Vakhitov–Kolokolov condition).

The regularity of NLS solitary waves is addressed in Appendix A.

## Notations

We denote the free Dirac operator by

$$D_m = D_0 + \beta m = -i\alpha \cdot \nabla + \beta m, \quad m > 0, \quad (1.3)$$

where  $D_0 = -i\alpha \cdot \nabla = -i \sum_{j=1}^n \alpha^j \frac{\partial}{\partial x^j}$ , with  $\alpha^j$  and  $\beta$  being self-adjoint  $N \times N$  Dirac matrices which satisfy

$$(\alpha^j)^2 = \beta^2 = I_N, \quad \alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta_{jk} I_N, \quad \alpha^j \beta + \beta \alpha^j = 0, \quad 1 \leq j, k \leq n.$$

$I_N$  is the  $N \times N$  identity matrix. The anticommutation relations lead to e.g.  $\text{Tr } \alpha^j = \text{Tr } \beta^{-1} \alpha^j \beta = -\text{Tr } \alpha^j = 0$ ,  $1 \leq j \leq n$ , and similarly  $\text{Tr } \beta = 0$ ; together with  $\sigma(\alpha^j) = \sigma(\beta) = \{\pm 1\}$ , this yields the conclusion that  $N$  is even.

For  $\psi \in \mathbb{C}^N$ , one denotes

$$\bar{\psi} = \psi^* \beta,$$

where  $\psi^*$  is the hermitian conjugate of  $\psi$ .

*Remark 1.1.* One can use the Clifford algebra representation theory (see e.g. [Fed96, Chapter 1, §5.3]) to show that there is a relation

$$N \in 2^{\lfloor \frac{n+1}{2} \rfloor} M, \quad M \in \mathbb{N}.$$

Without loss of generality, we may assume that the matrix  $\beta$  has the following form:

$$\beta = \begin{bmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{bmatrix}.$$

Then the anticommutation relations  $\{\alpha^j, \beta\} = 0$  show that the matrices  $(\alpha^j)_{1 \leq j \leq n}$  are block-antidiagonal,

$$\alpha^j = \begin{bmatrix} 0 & \sigma_j^* \\ \sigma_j & 0 \end{bmatrix}, \quad 1 \leq j \leq n,$$

where the matrices  $(\sigma_j)_{1 \leq j \leq n}$  satisfy

$$\sigma_j^* \sigma_k + \sigma_k^* \sigma_j = 2\delta_{jk}, \quad \sigma_j \sigma_k^* + \sigma_k \sigma_j^* = 2\delta_{jk}, \quad 1 \leq j, k \leq n. \quad (1.4)$$

*Remark 1.2.* The first relation in (1.4) implies the second one (and vice versa). Indeed, it was pointed out to us by A. Sukhtayev that the identity  $\sigma_j^* \sigma_j = \sigma_j \sigma_j^* = I_{N/2}$  allows us to turn the former relation in (1.4) into the latter multiplying it by  $\sigma_j$  from the left and by  $\sigma_j^*$  from the right.

*Remark 1.3.* It is well-known (see e.g. [KY01]) how to build the higher size Dirac matrices by induction; once we have  $n+1$  self-adjoint Dirac matrices  $\alpha^j$ ,  $1 \leq j \leq n$ , and  $\alpha_{n+1} := \beta$  in  $\mathbb{C}^N$ , then in  $\mathbb{C}^{2N}$  one has  $n+3$  self-adjoint Dirac matrices of the form

$$\begin{bmatrix} 0 & \alpha_j \\ \alpha_j & 0 \end{bmatrix}, \quad 1 \leq j \leq n+1, \quad \begin{bmatrix} 0 & -iI_N \\ iI_N & 0 \end{bmatrix}, \quad \begin{bmatrix} I_N & 0 \\ 0 & -I_N \end{bmatrix}.$$

This provides the possibility to choose  $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$ .

We denote  $r = |x|$  for  $x \in \mathbb{R}^n$ , and, abusing notations, we will also denote the operator of multiplication with  $|x|$  and  $\langle x \rangle = (1 + |x|^2)^{1/2}$  by  $r$  and  $\langle r \rangle$ , respectively.

The charge functional, (formally) conserved due to the  $U(1)$ -invariance of (2.1; NLDE), is denoted by  $Q$ :

$$Q(\psi) = \int_{\mathbb{R}^n} \psi^*(x, t) \psi(x, t) dx.$$

We denote the standard  $L^2$ -based Sobolev spaces of  $\mathbb{C}^N$ -valued functions by  $H^k(\mathbb{R}^n, \mathbb{C}^N)$ . For  $s, k \in \mathbb{R}$ , we define the weighted Sobolev spaces

$$H_s^k(\mathbb{R}^n, \mathbb{C}^N) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N), \|u\|_{H_s^k} < \infty \right\}, \quad \|u\|_{H_s^k} = \|\langle r \rangle^s \langle -i\nabla \rangle^k u\|_{L^2}.$$

We write  $L_s^2(\mathbb{R}^n, \mathbb{C}^N)$  for  $H_s^0(\mathbb{R}^n, \mathbb{C}^N)$ . For  $u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ , we denote  $\|u\| = \|u\|_{L^2}$ .

We will construct the solitary waves in the following Banach spaces:

$$X = L^2(\mathbb{R}, |t|^{n-1} dt; \mathbb{C}) \cap L^\infty(\mathbb{R}; \mathbb{C}), \quad \|\cdot\|_X = c \left( \|\cdot\|_{L^2(\mathbb{R}, |t|^{n-1} dt; \mathbb{C})} + \|\cdot\|_{L^\infty(\mathbb{R}; \mathbb{C})} \right), \quad (1.5)$$

$$X^1 = H^1(\mathbb{R}, \langle t \rangle^{n-1} dt; \mathbb{C}) = H_{(n-1)/2}^1(\mathbb{R}; \mathbb{C}) \subset X. \quad (1.6)$$

The space  $X^1$  is equipped with the standard norm of  $H_s^1(\mathbb{R})$ ,  $s = (n-1)/2$ , while the constant  $c > 0$  in (1.5) is chosen so that

$$\|\xi\|_X \leq \|\xi\|_{X^1}, \quad \forall \xi \in X^1. \quad (1.7)$$

We note that both  $X$  and  $X^1$  are algebras: there is  $C < \infty$  such that

$$\|\xi\eta\|_X \leq C \|\xi\|_X \|\eta\|_X, \quad \forall \xi, \eta \in X; \quad (1.8)$$

$$\|\xi\eta\|_{X^1} \leq C \|\xi\|_{X^1} \|\eta\|_{X^1}, \quad \forall \xi, \eta \in X^1. \quad (1.9)$$

Abusing notations, for  $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$  with  $\psi_1, \psi_2 \in X$ , we also denote

$$\|\psi\|_X = \sqrt{\|\psi_1\|_X^2 + \|\psi_2\|_X^2},$$

and similarly in the case of  $X^1$  instead of  $X$ .

The space

$$H_{e,o}^1(\mathbb{R}, |t|^{n-1} dt; \mathbb{C}^2) := H_{\text{even}}^1(\mathbb{R}, |t|^{n-1} dt; \mathbb{C}) \times H_{\text{odd}}^1(\mathbb{R}, |t|^{n-1} dt; \mathbb{C})$$

denotes the subspace of  $\mathbb{C}^2$ -valued functions on  $\mathbb{R}$  such that the first component is even and the second is odd. We also denote

$$X_{e,o} := L_{e,o}^2(\mathbb{R}, |t|^{n-1} dt; \mathbb{C}^2) \cap L^\infty(\mathbb{R}; \mathbb{C}^2), \quad X_{e,o}^1 := H_{e,o}^1(\mathbb{R}, \langle t \rangle^{n-1} dt; \mathbb{C}^2).$$

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## 2 Main results

We consider the nonlinear Dirac equation (1.1),

$$i\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \quad \psi(x, t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \quad (2.1; \text{NLDE})$$

where  $D_m$  is the Dirac operator (cf. (1.3)) and  $f \in C(\mathbb{R})$  with  $f(0) = 0$ . The structure of the nonlinearity is such that the equation is both  $\mathbf{U}(1)$ -invariant and hamiltonian, with the hamiltonian density given by

$$\mathcal{H}(\psi) = \psi^* D_m \psi - F(\psi^* \beta \psi),$$

with  $F(\tau) = \int_0^\tau f(t) dt$ ,  $\tau \in \mathbb{R}$ .

If  $\phi_\omega(x)e^{-i\omega t}$  is a solitary wave solution to (2.1; NLDE), then the profile  $\phi_\omega$  satisfies the stationary equation

$$\omega\phi_\omega = D_m\phi_\omega - f(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega. \quad (2.2)$$

In the nonrelativistic limit  $\omega \lesssim m$ , the solitary waves to nonlinear Dirac equation could be obtained as bifurcations of the solitary wave solutions  $\varphi_\omega(x)e^{-i\omega t}$  to the nonlinear Schrödinger equation

$$i\dot{\psi} = -\frac{1}{2m}\Delta\psi - |\psi|^{2k}\psi, \quad \psi(x, t) \in \mathbb{C}, \quad x \in \mathbb{R}^n. \quad (2.3)$$

By [Str77, BL83a] and [BGK83] (for the two-dimensional case), the stationary nonlinear Schrödinger equation

$$-\frac{1}{2m}u = -\frac{1}{2m}\Delta u - |u|^{2k}u, \quad u(x) \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 1 \quad (2.4)$$

has a strictly positive spherically symmetric exponentially decaying solution  $u_k \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$  (called the groundstate) if and only if  $0 < k < 2/(n-2)$  (any  $k > 0$  if  $n \leq 2$ ). The linearization at the solitary wave solution  $u_k(x)e^{-i\omega t}$  with  $\omega = -\frac{1}{2m}$  is given by  $\partial_t \mathbf{p} = \begin{bmatrix} 0 & \mathbf{l}_- \\ -\mathbf{l}_+ & 0 \end{bmatrix} \mathbf{p}$ ,  $\mathbf{p}(x, t) \in \mathbb{C}^2$ , where  $\mathbf{l}_\pm$  are defined by

$$\mathbf{l}_- = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^{2k}, \quad \mathbf{l}_+ = \frac{1}{2m} - \frac{\Delta}{2m} - (1+2k)u_k^{2k}. \quad (2.5)$$

By (2.4), the function  $u_{k,\lambda}(x) = \lambda^{1/k}u_k(\lambda x)$ ,  $\lambda > 0$ , satisfies the identity  $0 = \frac{\lambda^2}{2m}u_{k,\lambda} - \frac{1}{2m}\Delta u_{k,\lambda} - u_{k,\lambda}^{1+2k}$ . Differentiating this identity with respect to  $\lambda$  at  $\lambda = 1$  yields the following relation (which we will need in Lemma 7.1 below):

$$0 = \frac{1}{m}u_k + \mathbf{l}_+(\partial_\lambda|_{\lambda=1}u_{k,\lambda}) = \frac{1}{m}u_k + \mathbf{l}_+\left(\frac{1}{k}u_k + x \cdot \nabla u_k\right). \quad (2.6)$$

We set

$$\hat{V}(t) := u_k(|t|), \quad \hat{U}(t) := -\frac{1}{2m}\hat{V}'(t), \quad t \in \mathbb{R}, \quad (2.7)$$

where  $u_k$  is considered as a function of  $r = |x|$ ,  $x \in \mathbb{R}^n$ . Note that the inclusion  $u_k \in C^2(\mathbb{R}^n)$  implies that  $\hat{V} \in C^2(\mathbb{R})$  and  $\hat{U} \in C^1(\mathbb{R})$ . By (2.4), the functions  $\hat{V}$  and  $\hat{U}$  (which are even and odd, respectively) satisfy

$$\frac{1}{2m}\hat{V} + \partial_t\hat{U} + \frac{n-1}{t}\hat{U} = |\hat{V}|^{2k}\hat{V}, \quad \partial_t\hat{V} + 2m\hat{U} = 0, \quad t \in \mathbb{R}, \quad (2.8)$$

where  $\hat{U}(t)/t$  at  $t = 0$  is understood in the limit sense,  $\lim_{t \rightarrow 0} \hat{U}(t)/t = \hat{U}'(0)$ . We will obtain the solitary wave solutions to (2.1; NLDE) as bifurcations from  $(\hat{V}, \hat{U})$ .

**Theorem 2.1.** Let  $n \in \mathbb{N}$ ,  $N = 2^{[(n+1)/2]}$ , and assume that  $f \in C(\mathbb{R})$  and that there is  $k > 0$  such that

$$|f(\tau) - |\tau|^k| \leq o(|\tau|^k), \quad |\tau| \leq 1. \quad (2.9)$$

If  $n \geq 3$ , we additionally assume that  $k < 2/(n-2)$ .

1. There is

$$\omega_0 \in \left(\frac{m}{2}, m\right) \quad (2.10)$$

such that for all  $\omega \in (\omega_0, m)$  there are solitary wave solutions  $\phi_\omega(x)e^{-i\omega t}$  to (2.1; NLDE) with  $\phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$ , with

$$\phi_\omega(x) = \begin{bmatrix} v(r, \omega) \mathbf{n} \\ iu(r, \omega) (\mathbf{e}_r \cdot \boldsymbol{\sigma}) \mathbf{n} \end{bmatrix}, \quad r = |x|, \quad \mathbf{n} \in \mathbb{C}^{N/2}, \quad |\mathbf{n}| = 1, \quad (2.11)$$

$$\lim_{r \rightarrow 0} u(r, \omega) = 0. \quad (2.12)$$

Moreover if we express

$$v(r, \omega) = \epsilon^{\frac{1}{k}} V(\epsilon r, \epsilon), \quad u(r, \omega) = \epsilon^{1+\frac{1}{k}} U(\epsilon r, \epsilon); \quad \epsilon = \sqrt{m^2 - \omega^2} > 0, \quad r \geq 0, \quad (2.13)$$

decomposing

$$V(t, \epsilon) = \hat{V}(t) + \tilde{V}(t, \epsilon), \quad U(t, \epsilon) = \hat{U}(t) + \tilde{U}(t, \epsilon), \quad t \in \mathbb{R}, \quad \epsilon > 0, \quad (2.14)$$

with  $\hat{V}(t)$ ,  $\hat{U}(t)$  defined in (2.8), then there is  $\gamma > 0$  such that  $\tilde{V}(t, \epsilon)$ ,  $\tilde{U}(t, \epsilon)$  satisfy

$$\lim_{\epsilon \rightarrow 0+} \left\| e^{\gamma \langle t \rangle} \begin{bmatrix} \tilde{V}(\cdot, \epsilon) \\ \tilde{U}(\cdot, \epsilon) \end{bmatrix} \right\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = 0. \quad (2.15)$$

2. There is  $\epsilon_1 \in (0, \epsilon_0)$ ,  $\epsilon_0 := \sqrt{m^2 - \omega_0^2} > 0$ , such that

$$\epsilon_1 |U(t, \epsilon)| \leq \frac{1}{2} |V(t, \epsilon)|, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1), \quad (2.16)$$

$$\phi_\omega(x)^* \beta \phi_\omega(x) \geq |\phi_\omega(x)|^2 / 2, \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad \forall x \in \mathbb{R}^n, \quad \forall \epsilon \in (0, \epsilon_1). \quad (2.17)$$

3. One has

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq o(1) \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1), \quad (2.18)$$

where  $o(1)$  is with respect to  $\epsilon$  (so that  $o(1) \rightarrow 0$  as  $\epsilon \rightarrow 0$ ) uniformly in  $t$ , and there is  $b_0 < \infty$  such that

$$|V(t, \epsilon)| + |U(t, \epsilon)| \leq b_0 \langle t \rangle^{-(n-1)/2} e^{-|t|}, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1). \quad (2.19)$$

4. The solitary waves satisfy

$$\|\phi_\omega\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^N)} = O(\epsilon^{\frac{1}{k}}), \quad \|\phi_\omega\|_{L^2(\mathbb{R}^n, \mathbb{C}^N)} = O(\epsilon^{\frac{1}{k} - \frac{2}{n}}), \quad \omega \lesssim m. \quad (2.20)$$

5. Assume, moreover, that there is  $K > k$  such that

$$|f(\tau) - |\tau|^k| = O(|\tau|^K), \quad |\tau| \leq 1. \quad (2.21)$$

Then there are  $b_1, b_2 < \infty$  such that  $\tilde{V}(t, \epsilon), \tilde{U}(t, \epsilon)$  satisfy

$$\left\| e^{\gamma \langle t \rangle} \begin{bmatrix} \tilde{V}(\cdot, \epsilon) \\ \tilde{U}(\cdot, \epsilon) \end{bmatrix} \right\|_{H^1(\mathbb{R}, \mathbb{C}^2)} \leq b_1 \epsilon^{2\kappa}, \quad \epsilon \in (0, \epsilon_1) \quad (2.22)$$

and

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq b_2 \epsilon^{2\kappa} \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1), \quad (2.23)$$

with

$$\kappa = \min \left( 1, \frac{K}{k} - 1 \right). \quad (2.24)$$

**Remark 2.1.** We expect that, for solitary wave solutions  $\phi_\omega(x)e^{-i\omega t}$  to (2.1; NLDE), the profiles  $\phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$  are continuous and thus those of the form (2.11) always satisfy the condition (2.12).

Theorem 2.1 (I) is proved in Section 3. The positivity of  $\bar{\phi}\phi$  (Theorem 2.1 (2)) and the asymptotics of solitary waves (Theorem 2.1 (3)) are in Section 4. The asymptotics stated in Theorem 2.1 (4) follow from the estimates in Theorem 2.1 (I) and (2). The error estimates from Theorem 2.1 (5) are proved in Section 5.

**Theorem 2.2.** Let  $n \in \mathbb{N}$ ,  $N = 2^{[(n+1)/2]}$ , and assume that  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$  and that there are  $k > 0$  and  $K > k$  such that

$$|f(\tau) - |\tau|^k| = O(|\tau|^K), \quad |\tau| \leq 1; \quad (2.25)$$

$$|\tau f'(\tau) - k|\tau|^k| = O(|\tau|^K), \quad |\tau| \leq 1. \quad (2.26)$$

If  $n \geq 3$ , we additionally assume that  $k < 2/(n-2)$ . There is  $\epsilon_2 \in (0, \epsilon_1)$  small enough (with  $\epsilon_1 > 0$  from Theorem 2.1) so that for  $\omega = \sqrt{m^2 - \epsilon^2}$ ,  $\epsilon \in (0, \epsilon_2)$ , the functions  $\phi_\omega(x)$ ,  $\tilde{V}(t, \epsilon)$ , and  $\tilde{U}(t, \epsilon)$  from Theorem 2.1 (I) (cf. (2.11)–(2.15)) are unique and satisfy the following additional properties.

1. One has  $\phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^N)$ . The map  $\omega \mapsto \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$  is  $C^1$  (with  $\partial_\omega \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$ ). Moreover,  $\partial_\epsilon \tilde{W}(\cdot, \epsilon) \in H^1(\mathbb{R}, \mathbb{C}^2)$ , with

$$\|e^{\gamma \langle t \rangle} \partial_\epsilon \tilde{W}(\cdot, \epsilon)\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = O(\epsilon^{2\kappa-1}), \quad \epsilon \in (0, \epsilon_2), \quad (2.27)$$

where  $\tilde{W}(t, \epsilon) = \begin{bmatrix} \tilde{V}(t, \epsilon) \\ \tilde{U}(t, \epsilon) \end{bmatrix}$ , and there is  $c > 0$  such that

$$\|\partial_\omega \phi_\omega\|_{L^2}^2 = c\epsilon^{-n+\frac{2}{k}}(1 + O(\epsilon^{2\kappa})), \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad \epsilon \in (0, \epsilon_2). \quad (2.28)$$

2. Additionally, assume that  $k, K$  from (2.25) and (2.26) satisfy either

$$k < 2/n \quad (2.29)$$

or

$$k = 2/n, \quad K > 4/n. \quad (2.30)$$

Then there is  $\omega_* < m$  such that  $\partial_\omega Q(\omega) < 0$  for all  $\omega \in (\omega_*, m)$ .

If

$$k > 2/n, \quad (2.31)$$

then there is  $\omega_* < m$  such that  $\partial_\omega Q(\omega) > 0$  for all  $\omega \in (\omega_*, m)$ .



*Remark 2.2.* The absolute value in the expansion  $f(\tau) = |\tau|^k + \dots$  is needed in the case when  $k > 0$  is not an integer. We note that if  $k \in \mathbb{N}$  and is odd, then, with and without the absolute value, one arrives at two different models; for example, in the model

$$i\dot{\psi} = D_m\psi - (\bar{\psi}\psi)\beta\psi, \quad \psi(x, t) \in \mathbb{C}^4, \quad x \in \mathbb{R}^3,$$

where  $\bar{\psi} = \psi^*\beta$ , the small amplitude limit corresponding to  $\omega \rightarrow -m$  is a defocusing NLS (contrary to the small amplitude limit when  $\omega \rightarrow m$  which is a focusing NLS), while in the model

$$i\dot{\psi} = D_m\psi - |\bar{\psi}\psi|\beta\psi, \quad \psi(x, t) \in \mathbb{C}^4, \quad x \in \mathbb{R}^3,$$

such a limit is a focusing NLS (just like the small amplitude limit when  $\omega \rightarrow m$ ), and there is a symmetry  $\omega \rightarrow -\omega$  of solitary waves: if  $\phi(x)e^{-i\omega t}$  is a solitary wave, then so is  $\beta\gamma^5\phi(x)e^{i\omega t}$ , since  $\beta\gamma^5$  anticommutes with  $\alpha^j$ ,  $1 \leq j \leq 3$ , and with  $\beta$ . Above,  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$ .

More generally, in any dimension  $n \geq 1$ , if  $f(\tau) = f(|\tau|)$ , then, given the solitary wave solution  $\phi_\omega(x)e^{-i\omega t}$  with  $\phi_\omega(x)$  as in (2.11), there is also a solitary wave solution

$$\begin{bmatrix} u(r, \omega) (\mathbf{e}_r \cdot \boldsymbol{\sigma}^*) \mathbf{n} \\ -v(r, \omega) \mathbf{n} \end{bmatrix} e^{i\omega t}.$$

*Remark 2.3.* By [BL83b], the pure power stationary Schrödinger equation (2.4) with  $n \geq 3$  has infinitely many distinct radial solutions, and one expects that there is a family of solitary waves of (2.1; NLDE) bifurcating from any of these radial solutions (similarly to what we state in Theorem 2.1).

*Remark 2.4.* Let us also consider the following question: *Given a sequence of solitary wave solutions corresponding to  $\omega_j \rightarrow m$ , does this sequence (up to symmetries and extraction of a subsequence) always converge to a solution of a nonlinear Schrödinger equation, in the sense of the above lemma?* The answer to this question is negative in general. One obstacle can be illustrated as follows. In particular, in dimension  $n = 3$ , according to [ES95, Theorem 1], there are solitary wave solutions to (2.1; NLDE) for the pure power nonlinearity with  $k = 1$  or any real  $k \geq 2$  (so that  $|\tau|^{k+1}$  remains  $C^2$  at  $\tau = 0$ , meeting the assumptions of [ES95]); see also earlier works [CV86, Mer88]. On the other hand, if  $k \geq \frac{2}{n-2} = 2$ , the nonrelativistic limit can not converge to a stationary solution of the nonlinear Schrödinger equation, which does not exist for such values of  $k$ .

We do not know whether in the case  $k \leq \frac{2}{n-2}$ , any sequence of solitary waves  $\phi_\omega$ ,  $\omega \lesssim m$ , could be obtained as a bifurcation from an NLS solitary wave.

Theorem 2.2 (1) is proved in Section 6, and the Vakhitov–Kolokolov inequality in the critical case (Theorem 2.2 (2)) is analyzed in Section 7.

### 3 Solitary waves in the nonrelativistic limit. The case $f \in C$

In this section, we prove Theorem 2.1, constructing a particular family of solitary waves bifurcating from solitary waves of the nonlinear Schrödinger equation.

First of all, we need to rewrite the assumption  $f(\tau) = |\tau|^k + o(|\tau|^k)$  in a more convenient form. Fix  $k > 0$  (with  $k < 2/(n-2)$  if  $n \geq 3$ ). For  $\hat{V}, \hat{U}$  from (2.7), let us denote

$$\Lambda_k := \sup_{x \in \mathbb{R}^n} |\hat{V}(x)| + m \sup_{x \in \mathbb{R}^n} |\hat{U}(x)| < \infty. \quad (3.1)$$

We will focus on solitary waves with  $\tilde{V}(t, \epsilon), \tilde{U}(t, \epsilon)$  from (2.14) (we recall that  $\epsilon = \sqrt{m^2 - \omega^2}$ ), satisfying

$$|\tilde{V}(t, \epsilon)| + m|\tilde{U}(t, \epsilon)| < \Lambda_k, \quad \forall t \in \mathbb{R}; \quad (3.2)$$



we will see below that this will impose certain smallness assumptions onto  $\epsilon > 0$ . It follows from (3.1) and (3.2) that

$$|V(t, \epsilon)| \leq 2\Lambda_k, \quad m|U(t, \epsilon)| \leq 2\Lambda_k, \quad |V(t, \epsilon)^2 - \epsilon^2 U(t, \epsilon)^2| < 4\Lambda_k^2, \quad \forall t \in \mathbb{R}. \quad (3.3)$$

In the present analysis, we build small amplitude solitary waves, and the proof below would not be affected by a change of the nonlinearity  $f(\tau)$  outside of an open neighborhood of  $\tau = 0$ , hence, by (2.9), we could assume that

$$|f(\tau)| \leq 2|\tau|^k, \quad \tau \in \mathbb{R}, \quad (3.4)$$

and that

$$|f(\tau) - |\tau|^k| \leq |\tau|^k H(\tau), \quad \tau \in \mathbb{R}, \quad (3.5)$$

where  $H \in C(\mathbb{R})$  is monotonically increasing for  $\tau \geq 0$ , with  $H(0) = 0$ . It will be convenient for us to define

$$h(\epsilon) := \max \left( H(\epsilon^{2/k} 4\Lambda_k^2), \epsilon^{2k}, \epsilon^2 \right). \quad (3.6)$$

Note that, by (3.3),

$$H(v^2 - u^2) = H(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) \leq H(\epsilon^{2/k} 4\Lambda_k^2) \leq h(\epsilon); \quad (3.7)$$

from (3.5) and (3.7) we obtain the following convenient estimate for later use:

$$|f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 |V^2 - \epsilon^2 U^2|^k| \leq C\epsilon^2 |V^2 - \epsilon^2 U^2|^k h(\epsilon), \quad (3.8)$$

with  $h(\epsilon)$  continuous, monotonically increasing for  $\epsilon \geq 0$ , with  $h(0) = 0$ .

Now we are ready to start the proof of Theorem 2.1. We extend the argument of [CGG14, Section 4.2]. Substituting the Ansatz (2.11) into the nonlinear Dirac equation (2.1; NLDE) gives the system

$$\begin{cases} \partial_r u + \frac{n-1}{r} u + (m - \omega)v = f(v^2 - u^2)v, \\ \partial_r v + (m + \omega)u = f(v^2 - u^2)u, \end{cases} \quad r > 0, \quad (3.9)$$

for the pair of real-valued functions  $v = v(r, \omega)$ ,  $u = u(r, \omega)$ . We will always impose the condition

$$\lim_{r \rightarrow 0} u(r, \omega) = 0 \quad (3.10)$$

(cf. (2.12)); this allows us to extend  $v(r, \omega)$  and  $u(r, \omega)$  continuously onto  $\mathbb{R}$  so that  $v$  is even and  $u$  is odd:

$$v(r, \omega) = v(-r, \omega), \quad u(r, \omega) = -u(-r, \omega), \quad r \leq 0. \quad (3.11)$$

Then (3.9) extends onto the whole real axis:

$$\begin{cases} \partial_r u + \frac{n-1}{r} u + (m - \omega)v = f(v^2 - u^2)v, \\ \partial_r v + (m + \omega)u = f(v^2 - u^2)u, \\ u|_{r=0} = 0, \end{cases} \quad r \in \mathbb{R}. \quad (3.12)$$

In (3.12), the term  $\frac{u(r, \omega)}{r}$  at  $r = 0$  is understood as the limit  $\lim_{r \rightarrow 0} \frac{u(r, \omega)}{r} = \partial_r u(0, \omega)$ .

By (3.12),  $V$  and  $U$  from (2.13) are to satisfy

$$\begin{cases} \epsilon^2 \left( \partial_t U + \frac{n-1}{t} U \right) + (m - \omega)V = fV, \\ \partial_t V + (\omega + m)U = fU, \\ U|_{t=0} = 0, \end{cases} \quad t \in \mathbb{R},$$

with  $t = \epsilon r$  and with

$$f = f(\epsilon^{2/k}(V(t, \epsilon)^2 - \epsilon^2 U(t, \epsilon)^2)).$$

According to (3.11),  $V(t, \epsilon)$  is even in  $t \in \mathbb{R}$  and  $U(t, \epsilon)$  is odd. The term  $U/t$  at  $t = 0$  is understood as the limit  $\lim_{t \rightarrow 0} U(t, \epsilon)/t = \partial_t U(0, \epsilon)$ . We rewrite the above system as

$$\begin{cases} \partial_t U + \frac{n-1}{t}U + \frac{1}{m+\omega}V = \frac{f}{\epsilon^2}V, \\ \partial_t V + (m+\omega)U = fU, \\ U|_{t=0} = 0, \end{cases} \quad t \in \mathbb{R}. \quad (3.13)$$

We note that the system (2.8) corresponds to the limit of (3.13) as  $\epsilon \rightarrow 0$  (that is,  $\omega \rightarrow m$ ) after the substitution (2.14). For sufficiently small  $\epsilon > 0$ , we will construct the solution  $(V, U)$  as a bifurcation from  $(\hat{V}, \hat{U})$ .

Substituting  $V(t, \epsilon) = \hat{V}(t) + \tilde{V}(t, \epsilon)$  and  $U(t, \epsilon) = \hat{U}(t) + \tilde{U}(t, \epsilon)$  into (3.13) and then subtracting equations (2.8), we arrive at

$$\begin{cases} (\partial_t + \frac{n-1}{t})\tilde{U} + \frac{1}{m+\omega}\tilde{V} = (1+2k)|\hat{V}|^{2k}\tilde{V} - G_1(\epsilon, \tilde{V}, \tilde{U}), \\ \partial_t \tilde{V} + (m+\omega)\tilde{U} = G_2(\epsilon, \tilde{V}, \tilde{U}), \end{cases} \quad t \in \mathbb{R}, \quad \epsilon > 0, \quad (3.14)$$

where

$$G_1(\epsilon, \tilde{V}, \tilde{U}) = -\epsilon^{-2}f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2))V + \hat{V}^{2k}\hat{V} + (1+2k)\hat{V}^{2k}\tilde{V} + \left(\frac{1}{m+\omega} - \frac{1}{2m}\right)\hat{V}, \quad (3.15)$$

$$G_2(\epsilon, \tilde{V}, \tilde{U}) = f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2))U + (m-\omega)\hat{U}, \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad (3.16)$$

with (2.14) giving the relations between  $V, U$  and  $\tilde{V}, \tilde{U}$ . Let us denote

$$G(\epsilon, \tilde{W}) = \begin{bmatrix} G_1(\epsilon, \tilde{V}, \tilde{U}) \\ G_2(\epsilon, \tilde{V}, \tilde{U}) \end{bmatrix}, \quad \tilde{W} = \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix}, \quad (3.17)$$

and introduce the operator

$$A(\epsilon) = \begin{bmatrix} -\frac{1}{m+\omega} + (1+2k)|\hat{V}|^{2k} & -\partial_t - \frac{n-1}{t} \\ \partial_t & m+\omega \end{bmatrix}, \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad \epsilon \geq 0, \quad (3.18)$$

with the domain

$$D(A(\epsilon)) = H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2),$$

where

$$H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) := H_{\text{even}}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}) \times H_{\text{odd}}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C})$$

denotes the subspace of  $\mathbb{C}^2$ -valued functions on  $\mathbb{R}$  such that the first component is even and the second is odd. We similarly define the space  $L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$  and note that

$$A(\epsilon) : H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \rightarrow L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2).$$

Now the system (3.14) takes the form

$$A(\epsilon)\tilde{W}(t, \epsilon) = G(\epsilon, \tilde{W}(t, \epsilon)), \quad \epsilon > 0. \quad (3.19)$$

We notice that the differential operator  $A(\epsilon)$ ,  $\epsilon \in [0, m]$ , is self-adjoint on  $H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$ . We also notice that the essential spectrum of  $A(\epsilon)$ ,  $\epsilon \in [0, m]$ , with  $\hat{V}$  substituted by zero is  $(-\infty, -\frac{1}{m+\omega}] \cup [m +$

$\omega, +\infty)$  (see [Wei82, Satz 2.1] or [Tha92, Theorem 4.18]). Applying Weyl's criterion [RS78, Corollary 2 of Theorem XIII.14], we deduce that the essential spectrum of  $A(\epsilon)$  is also given by

$$\sigma_{\text{ess}}(A(\epsilon)) = \left(-\infty, -\frac{1}{m+\omega}\right] \cup [m+\omega, +\infty), \quad \epsilon \in [0, m].$$

Since the inclusion  $\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \ker A(0)$  would lead to  $\eta(t) = -\frac{1}{2m}\xi'(t)$  for  $t \in \mathbb{R}$  and then to  $\xi(|x|) \in \ker \mathfrak{l}_+$ , with  $\mathfrak{l}_+$  defined in (2.5) and  $x \in \mathbb{R}^n$ , while the restriction of  $\mathfrak{l}_+$  to spherically symmetric functions has zero kernel (see [CGNT08, Proof of Lemma 2.1, case  $k = 0$ ]), we see that  $\ker A(0)|_{H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)} = \{0\}$ . Thus,  $\lambda = 0$  does not belong to the spectrum of  $A(0)|_{L_{e,o}^2}$ , hence  $A(0)^{-1}$  is bounded from  $L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$  to  $H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$ . By continuity in  $\epsilon$  in the norm resolvent sense, there is  $\epsilon_0 > 0$  such that the mapping

$$A(\epsilon)^{-1} : L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \rightarrow H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2), \quad \epsilon \in [0, \epsilon_0] \quad (3.20)$$

is continuous, with the norm bounded uniformly in  $\epsilon \in [0, \epsilon_0]$ .

We actually need a stronger statement on continuity of  $A^{-1}$  in the following spaces (cf. (1.5), (1.6)):

$$X_{e,o} := L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \cap L^\infty(\mathbb{R}; \mathbb{C}^2), \quad X_{e,o}^1 := H_{e,o}^1(\mathbb{R}, \langle t \rangle^{n-1}dt; \mathbb{C}^2),$$

with the norms  $\|(\xi_1, \xi_2)\|_{X_{e,o}}^2 = \|\xi_1\|_X^2 + \|\xi_2\|_X^2$  for  $(\xi_1, \xi_2) \in X_{e,o}$  and  $\|(\xi_1, \xi_2)\|_{X_{e,o}^1}^2 = \|\xi_1\|_{X^1}^2 + \|\xi_2\|_{X^1}^2$  for  $(\xi_1, \xi_2) \in X_{e,o}^1$ . Abusing the notations, we will denote these norms by  $\|\cdot\|_X$  and  $\|\cdot\|_{X^1}$ , respectively.

**Lemma 3.1.** *The restriction of the mapping (3.20) to  $X_{e,o} \subset L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$  defines a continuous map*

$$A(\epsilon)^{-1} : X_{e,o} \rightarrow X_{e,o}^1, \quad \epsilon \in [0, \epsilon_0], \quad (3.21)$$

with the norm bounded uniformly in  $\epsilon \in [0, \epsilon_0]$ .

*Proof.* The uniform continuity in  $\epsilon$  will follow as in the previous case from the resolvent identity. Due to the continuity of the mapping (3.20), we already know that for any  $(b, a) \in L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \cap L^\infty(\mathbb{R}; \mathbb{C}^2)$  the solution of

$$A(\epsilon) \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} \in L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \cap L^\infty(\mathbb{R}; \mathbb{C}^2) \quad (3.22)$$

in  $L_{e,o}^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$  satisfies  $(v, u) \in H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$ .

In the case  $n = 1$ , we are done.

In the case  $n \geq 2$ , we proceed as follows. We already know that

$$(v, u) \in H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2) \subset H_{e,o}^1(\mathbb{R} \setminus [-1, 1], \langle t \rangle^{n-1}dt; \mathbb{C}^2).$$

It suffices to prove that  $(v, u)$  also satisfies

$$(v, u) \in L^\infty([-1, 1]; \mathbb{C}^2), \quad (v', u') \in L^\infty([-1, 1]; \mathbb{C}^2),$$

with the norms bounded by  $\|(a, b)\|_{L^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)} + \|(a, b)\|_{L^\infty(\mathbb{R}; \mathbb{C}^2)}$  (times a constant factor). Equation (3.22) can be written out as the following system:

$$\begin{cases} \left( (1+2k)\hat{V}(t)^{2k} - \frac{1}{m+\omega} \right) v(t) - \partial_t u - \frac{n-1}{t} u(t) = b(t), \\ \partial_t v + (m+\omega)u(t) = a(t). \end{cases} \quad (3.23)$$

From  $(v, u) \in H_{e,o}^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)$  we deduce that

$$v, u \in C(\mathbb{R} \setminus \{0\}),$$

and that  $|t|^{\frac{n-1}{2}}v \in L^\infty([-1, 1])$  and  $|t|^{\frac{n-1}{2}}u \in L^\infty([-1, 1])$ , as a consequence of Sobolev inequality and Hardy inequalities (see [Ste70, Appendix A.4 ( $r = 1$  and  $p = 2$ )] for the later) and moreover

$$\begin{aligned} \||t|^{\frac{n-1}{2}}v\|_{L^\infty([-1, 1])} + \||t|^{\frac{n-1}{2}}u\|_{L^\infty([-1, 1])} &\leq C\|(v, u)\|_{H^1(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)} \\ &\leq C'\|(b, a)\|_{L^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)}, \end{aligned} \quad (3.24)$$

with some  $C, C' < \infty$ .

We will proceed by induction; let us assume that, more generally,

$$\||t|^\alpha v\|_{L^\infty([-1, 1])} + \||t|^\alpha u\|_{L^\infty([-1, 1])} \leq C(\|(b, a)\|_{L^2(\mathbb{R}, |t|^{n-1}dt; \mathbb{C}^2)} + \|(b, a)\|_{L^\infty(\mathbb{R}; \mathbb{C}^2)}), \quad (3.25)$$

with  $C < \infty$  independent on  $(a(t), b(t))$  and with some  $\alpha \in [1/2, (n-1)/2]$ . Note that the upper bound is meaningless as  $t$  is bounded but we already know by (3.24) that (3.25) holds with  $\alpha = (n-1)/2$ . The first equation from (3.23) can be rewritten as

$$\partial_t(t^{n-1}u) = t^{n-1}\left((1+2k)\hat{V}(t)^{2k} - \frac{1}{m+\omega}\right)v(t) - t^{n-1}b(t). \quad (3.26)$$

Since  $u \in H^1(\mathbb{R}, |t|^{n-1}dt) \subset C(\mathbb{R} \setminus \{0\})$ ,  $|t|^{\frac{n-1}{2}}u \in L^\infty([-1, 1])$ , and  $n \geq 2$ , one has  $t^{n-1}u(t) \rightarrow 0$  as  $t \rightarrow 0$ ; therefore, integrating the relation (3.26), we arrive at

$$t^{n-1}u(t) = \int_0^t \left(s^{n-1}\left((1+2k)\hat{V}(s)^{2k} - \frac{1}{m+\omega}\right)v(s) - s^{n-1}b(s)\right) ds,$$

which yields

$$|t|^{n-1}|u(t)| \leq \left(\frac{C}{n-\alpha}|t|^{n-\alpha}\||t|^\alpha v\|_{L^\infty([-1, 1])} + \frac{1}{n}|t|^n\|b\|_{L^\infty([-1, 1])}\right), \quad t \in [-1, 1], \quad (3.27)$$

with  $C$  dependent on  $k$  and  $\hat{V}$  only; hence,

$$|u(t)| \leq \left(\frac{C}{n-\alpha}|t|^{1-\alpha}\||t|^\alpha v\|_{L^\infty([-1, 1])} + \frac{1}{n}|t|\|b\|_{L^\infty([-1, 1])}\right), \quad t \in [-1, 1]. \quad (3.28)$$

Similarly, from the second equation in (3.23) we deduce that

$$\||t|^\alpha \partial_t v\|_{L^\infty([-1, 1])} \leq |m+\omega|\||t|^\alpha u\|_{L^\infty([-1, 1])} + \||t|^\alpha a\|_{L^\infty([-1, 1])} =: C_*, \quad (3.29)$$

so that one has  $|v'(t)| \leq C_*|t|^{-\alpha}$ , therefore

$$|v(t)| \leq |v(1)| + |v(-1)| + C_*|t|^{-\alpha+1}/|\alpha-1|, \quad t \in [-1, 1] \setminus \{0\} \quad (3.30)$$

for  $\alpha \in \mathbb{R}_+ \setminus (\frac{1}{2}, \frac{3}{2})$  to have a uniform bound. For  $\alpha \in (1/2, 3/2)$ , we substitute  $\alpha$  in (3.29) with  $\alpha = 3/2$ , again arriving at (3.30).

To sum-up, given the estimates (3.25) on  $u$  and  $v$  with  $\alpha \in \{1/2\} \cup [3/2, +\infty)$ , the estimates (3.28) and (3.30) yield (3.25) with  $\max(\alpha-1, 0)$  in place of  $\alpha$ ; while given the estimates (3.25) with  $\alpha \in (1/2, 3/2)$ , we arrive at (3.25) with  $1/2$  in place of  $\alpha$ . It follows that (3.25) could be improved up to  $\alpha = 0$  in a finite number of steps. Having improved (3.25) to  $\alpha = 0$ , we use (3.27), (3.28) one more time, now with  $\alpha = 0$ , obtaining the bound

$$|u(t)/t| \leq \left(\frac{C}{n}\|v\|_{L^\infty([-1, 1])} + \frac{1}{n}\|b\|_{L^\infty([-1, 1])}\right), \quad t \in [-1, 1] \setminus \{0\}. \quad (3.31)$$

Using the resulting bounds on  $\|v\|_{L^\infty([-1, 1])}$  and  $\|u/t\|_{L^\infty([-1, 1])}$  in the system (3.23) yields the desired bounds on  $\|v'\|_{L^\infty([-1, 1])}$  and on  $\|u'\|_{L^\infty([-1, 1])}$ . The continuity of the mapping (3.21) is proved.  $\square$

*Remark 3.2.* We note that  $(v, u) \in X_{e,o}^1 \subset C(\mathbb{R}, \mathbb{C}^2)$ ; by (3.31), this implies that  $u(0) = 0$ .

The assumption  $(\tilde{V}, \tilde{U}) \in X_{e,o}^1 \subset X_{e,o}$  leads to  $(G_1(\tilde{V}, \tilde{U}), G_2(\tilde{V}, \tilde{U})) \in X_{e,o}$  (with  $G_1, G_2$  defined in (3.15) and (3.16)). Due to invertibility of  $A(\epsilon) : X_{e,o}^1 \rightarrow X_{e,o}$  (Lemma 3.1), the relation (3.19) leads to

$$\tilde{W} = A(\epsilon)^{-1}G(\epsilon, \tilde{W}), \quad \tilde{W} = \tilde{W}(t, \epsilon). \quad (3.32)$$

*Remark 3.3.* The continuity of  $f$  is not enough to conclude that the map

$$\mu : X_{e,o} \rightarrow X_{e,o}^1 \subset X_{e,o}, \quad \mu : \tilde{W} \mapsto A(\epsilon)^{-1}G(\epsilon, \tilde{W})$$

is a contraction, so we can not apply the contraction mapping principle to claim a unique fixed point of  $\mu$ ; we will retreat to the Schauder fixed point theorem instead, proving the existence of a fixed point but missing its uniqueness. In the case  $f \in C^1$ , indeed the mapping  $\mu$  can be shown to be a contraction on a particular subspace (see Lemma 6.3 below); this will allow us to prove uniqueness of a fixed point.

To be able to consider non-integer values of  $k > 0$  (in particular, we are going to treat the critical cases, when  $k = 2/n$ ), we need the following result.

**Lemma 3.4.** *For any  $k > 0$ , one has:*

$$\left| |a+b|^k - |a|^k \right| \leq 3^k \left( |a|^{k-\min(1,k)} + |b|^{k-\min(1,k)} \right) |b|^{\min(1,k)}, \quad a, b \in \mathbb{R}; \quad (3.33)$$

$$\left| |a+b|^k - |a|^k - k|a|^{k-1}b \operatorname{sgn} a \right| \leq 3^k \left( |a|^{k-\min(2,k)} + |b|^{k-\min(2,k)} \right) |b|^{\min(2,k)}, \quad a, b \in \mathbb{R}. \quad (3.34)$$

*Proof.* Since the inequalities (3.33) and (3.34) are homogeneous of degree  $k$  in  $a$  and  $b$ , it is enough to give a proof for  $a = 1, b \in \mathbb{R}$ .

If  $|b| \geq 1/2$ , then  $||1+b|^k - 1| \leq \max(|1+b|^k, 1) \leq 3^k|b|^k$ . If  $|b| < 1/2$ , then, by the mean value theorem,

$$\left| |1+b|^k - 1 \right| \leq \max_{c \in [1/2, 3/2]} k|c|^{k-1}|b|. \quad (3.35)$$

If  $k \geq 1$ , the right-hand side is bounded by  $k(3/2)^{k-1}|b| \leq 3^k|b|$  (since  $k(3/2)^{k-1} < 3^k, \forall k \in \mathbb{R}$ ). If  $k \in (0, 1)$ , the right-hand side of (3.35) is bounded by  $k2^{1-k}|b| = k|2b|^{1-k}|b|^k \leq k|b|^k \leq 3^k|b|^k$ . This completes the proof of (3.33).

Now let us prove (3.34); again, we only need to consider the case  $a = 1$ . For  $b \geq 1/2$ , one has

$$||1+b|^k - 1 - kb| \leq \max((3b)^k, 1 + kb) \leq \max(3^k, 2^k + 2^{k-1}k)b^k \leq 3^k(b^{k-\min(2,k)} + b^k).$$

In the last inequality, we took into account that, with  $b \geq 1/2$ ,

$$1 \leq 2^{\min(2,k)}b^{\min(2,k)} \leq 3^kb^{\min(2,k)}, \quad kb \leq k2^{k-1}b^k \leq 3^kb^k.$$

For  $b \leq -1/2$ , one similarly obtains

$$||1+b|^k - 1 - kb| \leq \max(|b|^k + k|b|, 1) \leq \max(1 + 2^{k-1}k, 2^k)|b|^k \leq 3^k|b|^k,$$

since  $1 + 2^{k-1}k < 3^k$  for  $k > 0$ .

Finally, for  $|b| < 1/2$ , by the mean value theorem,

$$\left| |1+b|^k - 1 - kb \right| \leq \max_{c \in [1/2, 3/2]} \frac{k|k-1|}{2} |c|^{k-2}|b|^2. \quad (3.36)$$

If  $k \geq 2$ , the right-hand side is bounded by  $\frac{1}{2}k(k-1)(3/2)^{k-2}|b|^2 \leq 3^k|b|^2$ , since  $\frac{1}{2}k|k-1|(3/2)^{k-2} < 3^k, \forall k > 0$ . If  $k \in (0, 2)$ , the right-hand side of (3.36) is bounded by

$$k|k-1|2^{2-k}|b|^2 = k|k-1||2b|^{2-k}|b|^k \leq k|k-1||b|^k \leq 3^k|b|^k. \quad \square$$

Recall that  $\Lambda_k < \infty$  was defined in (3.1).

**Lemma 3.5.** *There is  $C < \infty$  such that for any numbers  $\hat{V}, \hat{U}, \tilde{V}, \tilde{U} \in [-\Lambda_k, \Lambda_k]$ ,  $V = \hat{V} + \tilde{V}$ , and  $U = \hat{U} + \tilde{U}$ , one has*

$$\begin{aligned} |G_1(\epsilon, \tilde{V}, \tilde{U})| &\leq C \left( h(\epsilon)(|V| + |U|)^{1+2k} + \hat{V}^{1+2k-\min(2,1+2k)} |\tilde{V}|^{\min(2,1+2k)} + |\tilde{V}|^{1+2k} + \epsilon^2 \hat{V} \right), \\ |G_2(\epsilon, \tilde{V}, \tilde{U})| &\leq C \left( \epsilon^2 |V^2 - \epsilon^2 U^2|^k |U| + \epsilon^2 |\hat{U}| \right), \end{aligned}$$

for all  $\epsilon \in (0, \epsilon_0)$ , with  $\epsilon_0 > 0$  from Theorem 2.1

*Proof.* Although most terms in the definition of  $G$  (cf. (3.17)) are small, we have to be careful when we consider the general case  $k > 0$  when  $f'(\tau)$  may not be uniformly bounded near  $\tau = 0$ . To bound  $G_1$  (cf. (3.15)), we proceed as follows:

$$\begin{aligned} |G_1(\epsilon, \tilde{V}, \tilde{U})| &\leq \left| \epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) V - \hat{V}^{2k} \hat{V} - (1 + 2k) \hat{V}^{2k} \tilde{V} \right| + \left| \frac{\hat{V}}{m + \omega} - \frac{\hat{V}}{2m} \right| \\ &\leq \left| \epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - |V^2 - \epsilon^2 U^2|^k \right| |V| \\ &\quad + \left| |V^2 - \epsilon^2 U^2|^k - |V|^{2k} \right| |V| \\ &\quad + \left| |V|^{2k} V - \hat{V}^{2k} \hat{V} - (1 + 2k) \hat{V}^{2k} \tilde{V} \right| + \left( \frac{1}{m + \omega} - \frac{1}{2m} \right) \hat{V}. \end{aligned} \quad (3.37)$$

We use (3.8) to estimate the first term in the right-hand side by  $h(\epsilon)|V^2 - \epsilon^2 U^2|^k |V|$ . Other terms are dealt with by Lemma 3.4: we apply (3.33) to the second term and (3.34) (with  $1 + 2k$  instead of  $k$ ) to the third term, getting

$$\begin{aligned} |G_1(\epsilon, \tilde{V}, \tilde{U})| &\leq Ch(\epsilon) |V^2 - \epsilon^2 U^2|^k |V| + 3^k \left( |V|^{2k-2\min(1,k)} |\epsilon U|^{2\min(1,k)} + |\epsilon U|^{2k} \right) |V| \\ &\quad + 3^{1+2k} \left( \hat{V}^{1+2k-\min(2,1+2k)} |\tilde{V}|^{\min(2,1+2k)} + |\tilde{V}|^{1+2k} \right) + \left( \frac{1}{m + \omega} - \frac{1}{2m} \right) |\hat{V}|, \end{aligned}$$

which yields the desired bound on  $G_1$ . We took into account the definition of  $h(\epsilon)$  in (3.6).

The estimate on  $|G_2|$  immediately follows from (3.16) and (3.8).  $\square$

To apply the fixed point theorem, we will use the exponential weights, introducing compactness into (3.32). We fix

$$\gamma \in (0, \gamma_0), \quad \text{where} \quad \gamma_0 := \frac{1}{1 + 2k} \inf_{\epsilon \in [0, \epsilon_0]} \frac{1}{1 + \|A(\epsilon)^{-1}\|_{X_{e,o} \rightarrow X_{e,o}^1}}; \quad (3.38)$$

we note that, by Lemma 3.1, one has  $\gamma_0 > 0$ . Due to the exponential decay of  $\hat{V}(t), \hat{U}(t)$  (see Lemma A.1 in Appendix A), since  $\gamma < 1/(2k + 1) < 1$ , there are the following inclusions:

$$e^{(1+2k)\gamma\langle t \rangle} \hat{U} \in X^1, \quad e^{(1+2k)\gamma\langle t \rangle} \hat{V} \in X^1, \quad (3.39)$$

with  $X^1$  from (1.6). We define

$$A_\gamma(\epsilon) := e^{(1+2k)\gamma\langle t \rangle} \circ A(\epsilon) \circ e^{-(1+2k)\gamma\langle t \rangle} = A(\epsilon) - (1 + 2k)\gamma \frac{t}{\langle t \rangle} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (3.40)$$

Due to Lemma 3.1 and the choice of  $\gamma_0$  in (3.38), for any  $\epsilon \in [0, \epsilon_0]$  the operator (3.40) is closed and invertible, so that the mapping

$$A_\gamma(\epsilon)^{-1} = e^{(1+2k)\gamma\langle t \rangle} \circ A(\epsilon)^{-1} \circ e^{-(1+2k)\gamma\langle t \rangle} : X_{e,o} \rightarrow X_{e,o}^1 := H_{e,o}^1(\mathbb{R}, \langle t \rangle^{n-1} dt; \mathbb{C}^2) \quad (3.41)$$

is bounded uniformly in  $\epsilon \in [0, \epsilon_0]$ . We multiply the fixed point problem (3.32) by  $e^{\gamma\langle t \rangle}$ , rewriting it in the form

$$e^{\gamma\langle t \rangle} \tilde{W} = e^{-2k\gamma\langle t \rangle} A_\gamma(\epsilon)^{-1} e^{(1+2k)\gamma\langle t \rangle} G(\epsilon, e^{-\gamma\langle t \rangle} e^{\gamma\langle t \rangle} \tilde{W}). \quad (3.42)$$

**Lemma 3.6.** *There is  $C < \infty$  such that for any  $\epsilon \in (0, \epsilon_0)$  and any  $\begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \in X$  which satisfies*

$$\left\| e^{\gamma\langle t \rangle} \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\|_X \leq \left\| e^{\gamma\langle t \rangle} \begin{bmatrix} \hat{V}(t) \\ \hat{U}(t) \end{bmatrix} \right\|_X, \quad (3.43)$$

one has

$$\|e^{(1+2k)\gamma\langle t \rangle} G(\epsilon, \tilde{V}, \tilde{U})\|_X \leq C \left( h(\epsilon) + \left\| e^{\gamma\langle t \rangle} \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\|_X^{1+\min(1,2k)} \right), \quad (3.44)$$

with  $h(\epsilon)$  from (3.6).

*Proof.* We use the pointwise estimates on  $G_1, G_2$  from Lemma 3.5. There, the first term in the right-hand side of the bound on  $G_1$  has a factor  $h(\epsilon)$ . Multiplying this term by  $e^{(1+2k)\gamma\langle t \rangle}$  and using (3.39) and (3.43), and also the fact that the space  $X$  defined in (1.5) is closed under multiplication, we bound the resulting  $X$ -norm by  $Ch(\epsilon)$ , with some  $C < \infty$ .

The terms

$$\hat{V}^{1+2k-\min(2,1+2k)} |\tilde{V}|^{\min(2,1+2k)} + |\tilde{V}|^{1+2k}$$

in the right-hand side of the bound on  $G_1$  in Lemma 3.5, having no  $\epsilon$ -factor, are of order higher than one in  $\tilde{V}$ , benefiting us when  $|\tilde{V}|$  is small. Multiplying them by the factor  $e^{(1+2k)\gamma\langle t \rangle}$ , which is absorbed by the terms which are homogeneous of order  $(1+2k)$  in  $\hat{V}$  and  $\tilde{V}$ , we bound the  $X$ -norm of the result by  $C\|e^{\gamma\langle t \rangle} \tilde{V}\|_X^{1+\min(1,2k)}$ . We note that  $\|e^{\gamma\langle t \rangle} \hat{V}\|_X^{1+\min(1,2k)}$  and  $\|e^{\gamma\langle t \rangle} \hat{U}\|_X^{1+\min(1,2k)}$  are finite due to  $\gamma < 1$  (cf. (3.38)) and due to the exponential decay of  $\hat{V}$  and  $\hat{U}$  which follows from (2.7) and Lemma A.1 (see Appendix A).

For the last term in the right-hand side of the bound on  $G_1$  from Lemma 3.5 multiplied by  $e^{(1+2k)\gamma\langle t \rangle}$ , its  $X$ -norm is bounded by  $C\epsilon^2$  with the aid of (3.39). We conclude that there is a constant  $C < \infty$  such that there is the desired bound

$$\|e^{(1+2k)\gamma\langle t \rangle} G_1(\epsilon, \tilde{V}, \tilde{U})\|_X \leq C \left( h(\epsilon) + \|e^{\gamma\langle t \rangle} \tilde{V}\|_X^{1+\min(1,2k)} \right), \quad \forall \epsilon \in (0, \epsilon_0).$$

We now consider  $G_2$ . Due to the factor  $\epsilon^2$  in the right-hand side of the bound on  $G_2$  in Lemma 3.5 and due to the exponential decay of  $\hat{V}, \hat{U}$  (together with the bound (3.39)), as well as due to the assumption (3.43) about the exponential decay of  $\tilde{V}$  and  $\tilde{U}$ , one has

$$\|e^{(1+2k)\gamma\langle t \rangle} G_2(\epsilon, \tilde{V}, \tilde{U})\|_X \leq C\epsilon^2, \quad \forall \epsilon \in (0, \epsilon_0).$$

Lemma 3.6 is proved.  $\square$

Let us now complete the proof of Theorem 2.1. We consider the mapping

$$\begin{aligned} \mu_\gamma(\epsilon, \cdot) : X_{e,o} &\rightarrow X_{e,o} \rightarrow X_{e,o}^1 \subset X_{e,o}, \\ \mu_\gamma(\epsilon, \cdot) : Z &\mapsto e^{(1+2k)\gamma\langle t \rangle} G(\epsilon, e^{-\gamma\langle t \rangle} Z) \mapsto e^{-2k\gamma\langle t \rangle} A_\gamma(\epsilon)^{-1} e^{(1+2k)\gamma\langle t \rangle} G(\epsilon, e^{-\gamma\langle t \rangle} Z). \end{aligned} \quad (3.45)$$

Note that  $\tilde{W}$  is a solution to (3.19) if and only if  $Z = e^{\gamma\langle t \rangle} \tilde{W}$  is a fixed point of this map.



**Lemma 3.7.** *One can take  $\epsilon_0 > 0$  smaller if necessary so that there is  $a_0 > 0$  such that*

$$\mu_\gamma \left( \epsilon, \overline{\mathbb{B}_\rho(X_{e,o})} \right) \subset \overline{\mathbb{B}_\rho(X_{e,o}^1)}, \quad \rho = a_0 h(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0), \quad (3.46)$$

with  $h(\epsilon)$  from (3.6).

*Proof.* If  $Z$  belongs to a closed ball  $\overline{\mathbb{B}_\rho(X_{e,o})} = \{\xi \in X_{e,o} ; \|\xi\|_X \leq \rho\}$ , with

$$\rho \leq \left\| e^{\gamma \langle t \rangle} \begin{bmatrix} \hat{V}(t) \\ \hat{U}(t) \end{bmatrix} \right\|_X,$$

then Lemma 3.6 applies to  $\tilde{W} = e^{-\gamma \langle t \rangle} Z$ , giving us

$$\|e^{(1+2k)\gamma \langle t \rangle} G(\epsilon, e^{-\gamma \langle t \rangle} Z)\|_X \leq C \left\{ h(\epsilon) + \|Z\|_X^{1+\min(1,2k)} \right\}. \quad (3.47)$$

Therefore, to find the sufficient condition for (3.46) to be satisfied, we use the definition of  $\mu_\gamma$  from (3.45) and apply the estimate (3.47), arriving at the requirement

$$\|e^{-2k\gamma \langle t \rangle} \circ A_\gamma(\epsilon)^{-1}\|_{X_{e,o} \rightarrow X_{e,o}^1} C \left\{ h(\epsilon) + \rho^{1+\min(1,2k)} \right\} \leq \rho. \quad (3.48)$$

Noting the continuity of the mapping (3.41), the first factor in the left-hand side is bounded; thus, one can satisfy (3.48) by taking  $\rho = O(h(\epsilon))$ . This finishes the proof.  $\square$

Since it is not clear that the mapping  $\mu_\gamma(\epsilon, \cdot) : X_{e,o} \rightarrow X_{e,o}^1 \subset X_{e,o}$  defined in (3.45) is a contraction without assuming that  $f$  is sufficiently regular we can not apply the Banach fixed point theorem to (3.45). Instead, we use the Schauder fixed point theorem (see e.g. [GT01, Corollary 11.2]):

*Let  $Q$  be a closed, convex, bounded subset of a Banach space  $\mathcal{X}$  and  $\mu : Q \rightarrow Q$  a continuous compact map; then  $\mu$  has a fixed point in  $Q$ .*

Clearly, the mapping  $\mu_\gamma(\epsilon, \cdot) : X_{e,o} \rightarrow X_{e,o}^1$  is continuous; note that, in particular,

$$(V, U) \mapsto \epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) V$$

is continuous in the norm of the space  $X$  since the map  $(V, U) \mapsto \epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2))$  is continuous as a map from  $L^\infty(\mathbb{R}, \mathbb{C}^2)$  to  $L^\infty(\mathbb{R})$ . Then the mapping

$$e^{-2k\gamma \langle t \rangle} \circ A_\gamma(\epsilon)^{-1} : X_{e,o} \rightarrow X_{e,o}^1 \rightarrow X_{e,o},$$

is compact, since the multiplication by the decaying exponential weight is a compact map from  $X_{e,o}^1$  to  $X_{e,o}$ . Therefore, so is the mapping  $\mu_\gamma(\epsilon, \cdot)$  when considered as a map from  $X_{e,o}$  into itself. By Lemma 3.7, the Schauder fixed point theorem gives a fixed point of the map  $\mu_\gamma(\epsilon, \cdot)$  which belongs to a closed ball  $\overline{\mathbb{B}_\rho(X_{e,o}^1)}$  of radius  $\rho = a_0 h(\epsilon)$ , with  $a_0 > 0$  which does not depend on  $\epsilon \in (0, \epsilon_0)$ . It follows that  $\tilde{W} = e^{-k\gamma \langle t \rangle} Z$  satisfies

$$\|e^{k\gamma \langle t \rangle} \tilde{W}\|_{X^1} = \|Z\|_{X^1} \leq \rho \leq a_0 h(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0). \quad (3.49)$$

This yields (2.15).

*Remark 3.8.* The map  $\tilde{W}(\epsilon)$  is not a sufficiently well-defined function to make it continuous in  $\epsilon$  since the solution provided by the Schauder fixed point theorem is not necessarily unique, due to the absence of the contraction. The uniqueness of the mapping  $\epsilon \mapsto \tilde{W}(\epsilon)$ , under stronger assumptions on  $f$ , will be addressed in Section 6.2.

We note that

$$\|\tilde{W}\|_{L^\infty} \leq \|e^{k\gamma\langle t \rangle} \tilde{W}\|_{L^\infty} \leq \|e^{k\gamma\langle t \rangle} \tilde{W}\|_{X^1} \leq a_0 h(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0);$$

thus, we can impose the condition that  $\epsilon_0 > 0$  is small enough so that

$$|\tilde{V}(t, \epsilon)| + m|\tilde{U}(t, \epsilon)| < \Lambda_k, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_0),$$

to satisfy our assumption (3.2).

Finally, let us prove that  $V, U \in C^1(\mathbb{R})$ . Due to the continuity of  $\hat{V}$  and  $\hat{U}$  (which follows from Lemma A.1 and from (2.7)) and of  $\tilde{V}$  and  $\tilde{U}$  (which follows from applying Lemma 3.1 to (3.19)), we know that  $V$  and  $U$  are continuous on the whole real axis.

**Lemma 3.9.** *Fix  $\epsilon \in (0, \epsilon_0)$ . If  $f \in C(\mathbb{R})$  and if  $V, U \in C(\mathbb{R})$ , with  $V$  even and  $U$  odd, are solutions to (3.13), then  $V, U \in C^1(\mathbb{R})$  and  $H(t) := U(t)/t$ ,  $t \neq 0$  could be extended to a continuous function on  $\mathbb{R}$ .*

*Moreover, if there is  $C < \infty$  such that*

$$|V(t)| + |U(t)| \leq C, \quad \forall t \in \mathbb{R},$$

*then there is  $C' < \infty$  such that*

$$|\partial_t V(t)| + |\partial_t U(t)| \leq C', \quad \forall t \in \mathbb{R}.$$

*Proof.* The second equation in (3.13) immediately gives  $V \in C^1(\mathbb{R})$ . To prove that one also has  $U \in C^1(\mathbb{R})$ , we write the first equation in (3.13) as

$$U' + (n-1)\frac{U}{t} = B(t), \quad t \in \mathbb{R}, \quad (3.50)$$

with  $B \in C(\mathbb{R})$  given by

$$B(t) = \frac{f(\epsilon^{2/k}(V(t)^2 - \epsilon^2 U(t)^2))}{\epsilon^2} V(t) - \frac{1}{m+\omega} V(t). \quad (3.51)$$

It is enough to prove that  $H(t) = U(t)/t \in C(\mathbb{R} \setminus \{0\})$  could be extended to a continuous function on  $\mathbb{R}$  (then the same is true for  $U'$ ). Thus, we need to show that  $H(t)$  has a finite limit as  $t \rightarrow 0$ . From (3.50) we arrive at

$$\partial_t(U(t)t^{n-1}) = B(t)t^{n-1}, \quad t \in \mathbb{R},$$

hence, one has

$$H(t) = \frac{U(t)}{t} = \frac{\int_0^t B(\tau)\tau^{n-1} d\tau}{t^n}, \quad t > 0, \quad (3.52)$$

which has a well-defined limit at the origin:  $\lim_{t \rightarrow 0} H(t) = \lim_{t \rightarrow 0} \frac{\int_0^t B(\tau)\tau^{n-1} d\tau}{t^n} = \lim_{t \rightarrow 0} B(t)/n = B(0)/n$ .

Let us show the uniform boundedness of the derivatives of  $V$  and  $U$ . From the system (3.13), due to bounds (3.3), we conclude that

$$|\partial_t V(t)| \leq C, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_0),$$

with some  $C < \infty$ . Then, since  $B(t)$  in (3.51) satisfies

$$|B(t)| = \left| \frac{f}{\epsilon^2} V - \frac{1}{m+\omega} U \right| \leq C, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_0),$$

with some  $C < \infty$ , we conclude from (3.52) that  $|H(t)| \leq \|B\|_{L^\infty}$  and then from (3.13) that  $|\partial_t U(t)| \leq 2\|B\|_{L^\infty}$ , for all  $t \in \mathbb{R}$ .  $\square$

The proof of Theorem 2.1 (I) is finished.

## 4 Positivity of $\bar{\phi}\phi$ and improved estimates

### 4.1 Positivity of $\bar{\phi}\phi$ in the nonrelativistic limit via the shooting argument

To be able to consider the nonlinearity  $f(\tau) = |\tau|^k + \dots$  which is not differentiable at  $\tau = 0$  unless  $k \geq 1$ , we will show that the quantity  $\phi^* \beta \phi$ , which is the argument of  $f(\cdot)$  in (2.2), remains positive if  $\omega \lesssim m$ . This will allow us to treat the nonlinear Dirac equation with fractional power nonlinearity using the Taylor-style estimates on the remainders instead of weaker estimates from Lemma 3.4.

So we proceed to the proof of Theorem 2.1 (2), showing that  $U$  is pointwise dominated by  $V$ .

**Proposition 4.1.** *There is  $\epsilon_1 \in (0, \epsilon_0)$  such that for all  $\epsilon \in (0, \epsilon_1)$  one has*

$$\epsilon_1 |U(t, \epsilon)| \leq \frac{1}{2} |V(t, \epsilon)|, \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1).$$

Above,  $\epsilon_0 > 0$  is from Theorem 2.1 (1).

*Proof.* We rewrite (3.13) as follows:

$$\begin{cases} \partial_t U = -\frac{1}{m+\omega} V - \frac{n-1}{t} U + |V|^{2k} V + (\epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - |V|^{2k}) V, \\ \partial_t V = -(m+\omega) U + \epsilon^2 |V|^{2k} U + (f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 |V|^{2k}) U. \end{cases} \quad (4.1)$$

For any  $\delta > 0$  and any  $\nu \in (0, \nu_0)$ ,  $\nu_0 = \min(\delta/8, m\delta/8)$ , define the following closed sets (see Figure 1):

$$\begin{aligned} \mathcal{K}_{\delta, \nu}^+ &= \left\{ (V, U) \in \overline{\mathbb{B}_\delta^2} \subset \mathbb{R}^2 ; U \geq \max\left(0, \frac{V+\nu}{m}, \frac{2V}{m}\right) \right\}, \\ \mathcal{K}_\delta^0 &= \left\{ (V, U) \in \overline{\mathbb{B}_\delta^2} \subset \mathbb{R}^2 ; V \geq 0, \frac{V}{4m} \leq U \leq \frac{2V}{m} \right\}, \\ \mathcal{K}_{\delta, \nu}^- &= \left\{ (V, U) \in \overline{\mathbb{B}_\delta^2} \subset \mathbb{R}^2 ; V \geq 0, U \leq \min\left(\frac{V-\nu}{2m}, \frac{V}{4m}\right) \right\}. \end{aligned}$$

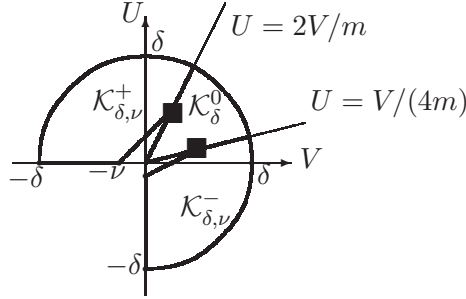


Figure 1: The regions  $\mathcal{K}_{\delta, \nu}^+$ ,  $\mathcal{K}_\delta^0$ ,  $\mathcal{K}_{\delta, \nu}^-$  inside  $\overline{\mathbb{B}_\delta^2}$ .

The value of  $\nu_0$  is chosen so that for  $\nu \in (0, \nu_0)$  the corner points of both  $\mathcal{K}_{\delta, \nu}^+$  and  $\mathcal{K}_{\delta, \nu}^-$  inside the first quadrant,  $(\nu, 2\nu/m)$  and  $(2\nu, \nu/(2m))$  (marked by black squares on Figure 1), belong to  $\mathbb{B}_{\delta/2}^2$ :

$$(\nu, 2\nu/m) \in \mathbb{B}_{\delta/2}^2, \quad (2\nu, \nu/(2m)) \in \mathbb{B}_{\delta/2}^2. \quad (4.2)$$

**Lemma 4.2.** *If  $\delta > 0$  is sufficiently small, then any  $C^1$ -solution to (4.1) with  $\epsilon \in (0, \epsilon_0)$  (with  $\epsilon_0 > 0$  from Theorem 2.1) which satisfies*

$$(V(T), U(T)) \in \mathcal{K}_{\delta, \nu}^+$$

*at some  $T \geq 2n$ , can only leave the region  $\mathcal{K}_{\delta, \nu}^+$  through the boundary of the  $\delta$ -disc: either  $(V(t), U(t)) \in \mathcal{K}_{\delta, \nu}^+$  for all  $t \geq T$ , or else there is  $T_* \in (T, +\infty)$  such that  $(V(t), U(t)) \in \mathcal{K}_{\delta, \nu}^+$  for  $T \leq t \leq T_*$ ,  $(V(T_*), U(T_*)) \in \mathbb{S}_\delta^1$ .*

*Proof.* It suffices to check that at all pieces of  $\partial\mathcal{K}_{\delta,\nu}^+ \setminus \mathbb{S}_\delta^1$  the integral curves of (4.1) are directed strictly inside  $\mathcal{K}_{\delta,\nu}^+$ ; that is, at the points  $U = \max(0, (V + \nu)/m, 2V/m)$ , one has  $\mathbf{n} \cdot (\dot{V}, \dot{U}) > 0$ , with  $\mathbf{n}$  the inner normal to  $\partial\mathcal{K}_{\delta,\nu}^+$  (as long as  $t \geq 2n$ ).

On the piece  $\{(V, 0) ; -\delta \leq V \leq -\nu\} \subset \partial\mathcal{K}_{\delta,\nu}^+$ , we compute:

$$(0, 1) \cdot (\dot{V}, \dot{U}) = \dot{U} = -\frac{V}{m + \omega} + o(V) > 0,$$

as long as  $\delta > 0$  is sufficiently small.

On the piece  $\{(V, (V + \nu)/m) ; -\nu \leq V \leq \nu\} \subset \partial\mathcal{K}_{\delta,\nu}^+$ , since  $T \geq 2n$ , one has:

$$\begin{aligned} (-1, m) \cdot (\dot{V}, \dot{U}) &= (m + \omega)U - m\left(\frac{V}{m + \omega} + \frac{(n-1)U}{t}\right) + o(|U| + |V|) \\ &\geq \left(\frac{m}{2} + \omega\right)U - \frac{mV}{m + \omega} + o(|U| + |V|) = \left(\frac{1}{2} + \frac{\omega}{m}\right)(V + \nu) - \frac{mV}{m + \omega} + o(|V + \nu| + |V|). \end{aligned}$$

When  $-\nu \leq V < 0$ , the first two terms in the right-hand side are positive, dominating the last term if  $\delta$  is sufficiently small. For  $0 \leq V \leq \nu$ , due to  $\omega > m/2$  (cf. (2.10)), the positive first term in the right-hand side dominates both the second term and the last term since

$$\frac{mV}{m + \omega} \leq \frac{2}{3}V \leq \frac{1}{3}(V + \nu).$$

On the piece of the boundary  $\{(V, 2V/m) ; V \geq \nu\} \cap \partial\mathcal{K}_{\delta,\nu}^+$ , we get

$$\begin{aligned} (-2, m) \cdot (\dot{V}, \dot{U}) &= -2\dot{V} + m\dot{U} = 2(m + \omega)U - m\left(\frac{V}{m + \omega} + \frac{n-1}{t}U\right) + o(V) \\ &\geq 2(m + \omega)\frac{2V}{m} - \left(V + \frac{n-1}{t}2V\right) + o(V) \geq 4V + o(V) > 0. \end{aligned}$$

We took into account that  $\omega > m/2$  and that  $t \geq 2n$ . □

**Lemma 4.3.** *If  $\delta > 0$  is sufficiently small, then any  $C^1$ -solution to (4.1) with  $0 < \epsilon \leq \frac{m}{4}$  which satisfies*

$$(V(T), U(T)) \in \mathcal{K}_{\delta,\nu}^-$$

*at some  $T \geq 2n$  can only exit the region  $\mathcal{K}_{\delta,\nu}^-$  through the boundary of the  $\delta$ -disc: either  $(V(t), U(t)) \in \mathcal{K}_{\delta,\nu}^-$  for all  $t \geq T$ , or else there is  $T_* \in (T, +\infty)$  such that  $(V(t), U(t)) \in \mathcal{K}_{\delta,\nu}^-$  for  $T \leq t \leq T_*$ ,  $(V(T_*), U(T_*)) \in \mathbb{S}_\delta^1$ .*

*Proof.* The proof is similar to that of Lemma 4.3; we keep checking the positivity of the dot products of the inner normals to the boundary with  $(\dot{V}, \dot{U})$ . For the pieces of the boundary given by  $V = 0$ , the proof is immediate (from (4.1), one can see that  $\dot{V} > 0$ , as long as  $\delta > 0$  is small enough so that the nonlinear terms are dominated by the linear part). On the piece given by  $U = (V - \nu)/(2m)$ ,  $0 \leq V \leq 2\nu$ ,

$$(1, -2m) \cdot (\dot{V}, \dot{U}) = \dot{V} - 2m\dot{U} = -(m + \omega)U + \frac{2mV}{m + \omega} + \frac{2m(n-1)U}{t} + o(|V| + |U|). \quad (4.3)$$

At  $V = 0$ ,  $U = -\nu/(2m)$ , the linear part of the right-hand side of (4.3) equals  $\frac{m+\omega}{2m}\nu - \frac{n-1}{t}\nu$ , which is positive for  $\nu > 0$ ,  $\omega \in (0, m)$ ,  $t \geq 2n$ . At the other end of the interval, at  $V = 2\nu$ ,  $U = \nu/(2m)$ , the linear part of (4.3) equals  $-\frac{m+\omega}{2m}\nu + \frac{4m}{m+\omega}\nu + \frac{n-1}{t}\nu$ , which is strictly positive for  $t \geq 2n$ ,  $\nu > 0$ ,  $\omega \in (m/2, m)$ . Since the linear part is strictly positive, it dominates the error term  $o(|V| + |U|)$  in (4.3) as long as  $\delta > 0$  is sufficiently small.

On the piece of the boundary of  $\mathcal{K}_{\delta,\nu}^-$  given by  $U = V/(4m)$ ,  $2\nu \leq V \leq \delta$ , one has

$$\begin{aligned} (1, -4m) \cdot (\dot{V}, \dot{U}) &= \dot{V} - 4m\dot{U} = -(m + \omega)U + \frac{4mV}{m + \omega} + \frac{4m(n-1)U}{t} + o(|V| + |U|) \\ &= \left( -\frac{m + \omega}{4m} + \frac{4m}{m + \omega} + \frac{n-1}{t} \right) V + o(|V|). \end{aligned}$$

Since  $\omega \in (m/2, m)$  (cf. (2.10)), the linear part in the right-hand side is strictly positive, dominating the nonlinear part as long as  $\delta > 0$  is sufficiently small.  $\square$

Back to the proof of the proposition, we choose  $\delta > 0$  small enough so that both Lemma 4.2 and Lemma 4.3 are satisfied. By [BL83a],  $\hat{V} > 0$  and  $\hat{U} \geq 0$  are exponentially decaying, hence we can choose  $T_1$  large enough and take  $\delta > 0$  smaller if necessary so that

$$T_1 \geq 2n, \quad (\hat{V}(T_1), \hat{U}(T_1)) \in Q_\delta := (\mathbb{B}_{3\delta/4}^2 \setminus \mathbb{B}_{2\delta/3}^2) \cap \{(V, U) ; V \geq 0, U \geq 0\}, \quad (4.4)$$

and so that

$$(\hat{V}(t), \hat{U}(t)) \in \mathbb{B}_{3\delta/4}^2, \quad \forall t \geq T_1. \quad (4.5)$$

By (3.49),

$$\|\tilde{V}(\cdot, \epsilon)\|_{L^\infty} + \|\tilde{U}(\cdot, \epsilon)\|_{L^\infty} = O(h(\epsilon)). \quad (4.6)$$

Since  $Q_\delta$  is strictly inside  $\mathcal{K}_{\delta,\nu}^+ \cup \mathcal{K}_\delta^0 \cup \mathcal{K}_{\delta,\nu}^-$  (this is due to choosing  $\nu_0 > 0$  such that (4.2) is satisfied for  $\nu \in (0, \nu_0)$ ), we use (4.4) and (4.6) to conclude that there is  $\epsilon_1 \in (0, \epsilon_0)$  such that

$$(V(T_1, \epsilon), U(T_1, \epsilon)) = (\hat{V}(T_1) + \tilde{V}(T_1, \epsilon), \hat{U}(T_1) + \tilde{U}(T_1, \epsilon)) \in \mathcal{K}_{\delta,\nu}^+ \cup \mathcal{K}_\delta^0 \cup \mathcal{K}_{\delta,\nu}^-, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.7)$$

Moreover, by (4.5) and (4.6), we could take

$$\epsilon_1 \in (0, \epsilon_0)$$

smaller if necessary so that

$$(V(t, \epsilon), U(t, \epsilon)) = (\hat{V}(t) + \tilde{V}(t, \epsilon), \hat{U}(t) + \tilde{U}(t, \epsilon)) \in \mathbb{B}_\delta^2, \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.8)$$

**Lemma 4.4.** *One has*

$$V(t, \epsilon) > 0, \quad U(t, \epsilon) > 0, \quad \frac{V(t, \epsilon)}{4m} < U(t, \epsilon) < \frac{2V(t, \epsilon)}{m}, \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1).$$

*Proof.* We claim that the solution  $(V(t, \epsilon), U(t, \epsilon))$  stays in  $\mathcal{K}_\delta^0$  for all  $t \geq T_1$ .

First, we notice that if  $(V(T_1, \epsilon), U(T_1, \epsilon)) \in \mathcal{K}_\delta^0$ , then for  $t \geq T_1$  the trajectory  $(V(t), U(t))$  could not leave  $\mathcal{K}_\delta^0$  through the arc of the  $\delta$ -circle in the first quadrant (due to (4.8)). At the same time, it can not leave  $\mathcal{K}_\delta^0$  through  $(V, U) = (0, 0) \in \mathcal{K}_\delta^0$  because of the uniqueness of the solution passing through  $(0, 0)$  (for  $t \geq T_1 \geq 2n$ , the right-hand side of the system (4.1) is Lipschitz in  $(V, U) \in \mathcal{K}_\delta^0$ ; this unique solution is  $V(t) \equiv U(t) \equiv 0$ ,  $t \geq T_1$ ).

The solution also could not leave  $\mathcal{K}_\delta^0$  through the side  $U = 2V/m$  (with  $V > 0$ ). Indeed, the assumption that  $U(T_*, \epsilon) = 2V(T_*, \epsilon)/m > 0$  at some  $T_* \geq T_1$  leads to a contradiction: we choose  $\nu > 0$  small enough (one can take  $\nu = \min(\nu_0, V(T_*, \epsilon)) > 0$ ) so that  $(V(T_*, \epsilon), U(T_*, \epsilon)) \in \mathcal{K}_{\delta,\nu}^+$ , and then Lemma 4.2 together with the bound (4.8) show that the solution would be trapped in  $\mathcal{K}_{\delta,\nu}^+$  for all  $t \geq T_1$ , hence would not be able to converge to zero as  $t \rightarrow \infty$ . For the same reason, the solution can not start in this region initially, at  $t = T_1$ : one should have  $(V(T_1, \epsilon), U(T_1, \epsilon)) \notin \mathcal{K}_{\delta,\nu}^+$  for any  $\nu \in (0, \nu_0]$ .

The same argument (now with the aid of Lemma 4.3) shows that one can not have  $U = V/(4m)$ ,  $V > 0$  at some  $T_* \geq T_1$ , neither can the solution start at  $t = T_1$  in  $\mathcal{K}_{\delta,\nu}^-$  for any  $\nu \in (0, \nu_0]$ : the solution  $(V(t, \epsilon), U(t, \epsilon))$  would be trapped in  $\mathcal{K}_{\delta,\nu}^-$  for all  $t \geq T_1$  and thus could not converge to zero.

Thus, by (4.7), the trajectory  $(V(t, \epsilon), U(t, \epsilon))$  starts strictly inside  $\mathcal{K}_\delta^0$  at  $t = T_1$  and stays there for all  $t \geq T_1$ . The statement of the lemma follows.  $\square$

Due to  $V$  being even and  $U$  being odd in  $t$ , Lemma 4.4 also yields the inequality

$$|U(t, \epsilon)| < \frac{2}{m} V(t, \epsilon), \quad |t| \geq T_1, \quad \epsilon \in (0, \epsilon_1). \quad (4.9)$$

Let us now consider the case  $|t| \leq T_1$ . By (3.49), there is  $C > 0$  such that

$$\sup_{|t| \leq T_1} |U(t, \epsilon)| \leq \sup_{|t| \leq T_1} \hat{U}(t) + \|\tilde{U}(\cdot, \epsilon)\|_{L^\infty} \leq \sup_{|t| \leq T_1} \hat{U}(t) + Ch(\epsilon); \quad (4.10)$$

on the other hand, again using (3.49), we have, for all  $\epsilon \in (0, \epsilon_1)$ :

$$\inf_{|t| \leq T_1} V(t, \epsilon) \geq \inf_{|t| \leq T_1} \hat{V}(t) - \|\tilde{V}(\cdot, \epsilon)\|_{L^\infty} \geq \inf_{|t| \leq T_1} \hat{V}(t) - Ch(\epsilon) \geq \inf_{|t| \leq T_1} \hat{V}(t)/2 > 0 \quad (4.11)$$

if we choose  $\epsilon_1 > 0$  is so small that  $Ch(\epsilon_1) < \inf_{|t| \leq T_1} \hat{V}(t)/2$ . It follows from (4.10) and (4.11) that for some  $C' < \infty$  we could write

$$|U(t, \epsilon)| < C' V(t, \epsilon), \quad |t| \leq T_1, \quad \epsilon \in (0, \epsilon_1). \quad (4.12)$$

We require that  $\epsilon_1 > 0$  be small enough, satisfying  $\epsilon_1 \leq \min(m/2, 1/(2C'))$ ; then the inequalities (4.9) and (4.12) yield (2.16), finishing the proof of Proposition 4.1.  $\square$

Using the inequality (2.16), one derives the bound (2.17):

$$\phi_\omega^* \beta \phi_\omega = v^2 - u^2 = \epsilon^{\frac{2}{k}} (V^2 - \epsilon^2 U^2) \geq \epsilon^{\frac{2}{k}} \frac{3V^2}{4} \geq \epsilon^{\frac{2}{k}} \frac{2V^2 + 2\epsilon^2 U^2}{4} = \frac{\phi_\omega^* \phi_\omega}{2}, \quad \omega \in (\omega_1, m),$$

with  $\omega_1 = \sqrt{m^2 - \epsilon_1^2}$ . This completes the proof of Theorem 2.1 (2).

## 4.2 Sharp decay asymptotics and optimal estimates

We now prove Theorem 2.1 (3). We will derive the sharp exponential decay of each of  $V, \hat{V}, U, \hat{U}$  and then prove that, as the matter of fact,  $\tilde{V}$  and  $\tilde{U}$  are pointwise dominated by  $V$ . We recall that  $\hat{V}$  and  $\hat{U}$  are obtained from NLS solitary waves and that

$$V(t, \epsilon) = \hat{V}(t) + \tilde{V}(t, \epsilon), \quad U(t, \epsilon) = \hat{U}(t) + \tilde{U}(t, \epsilon);$$

cf. (2.7), (2.14).

**Lemma 4.5.** *There are  $C_1 > c_1 > 0$  such that for all  $\epsilon \in (0, \epsilon_1)$  one has*

$$|V(t, \epsilon)| \geq c_1 t^{-(n-1)/2} e^{-t}, \quad |V(t, \epsilon)| + |U(t, \epsilon)| \leq C_1 t^{-(n-1)/2} e^{-t}, \quad \forall t \geq T_1; \quad (4.13)$$

$$\hat{V}(t) \geq c_1 t^{-(n-1)/2} e^{-t}, \quad \hat{V}(t) + |\hat{U}(t)| \leq C_1 t^{-(n-1)/2} e^{-t}, \quad \forall t \geq T_1; \quad (4.14)$$

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq C_1 t^{-(n-1)/2} e^{-t}, \quad \forall t \geq T_1. \quad (4.15)$$

Above,  $\epsilon_1 > 0$  is from Theorem 2.1 (2) and  $T_1 < \infty$  is from (4.4).

*Proof.* The inequality (4.15) follows from (4.13) and (4.14).

The inequalities (4.13) and (4.14) are proved similarly. We will focus on (4.13), which are more involved; then the inequalities (4.14) could be obtained by taking the limit  $\epsilon \rightarrow 0$ .

We introduce  $\mathcal{V}(t, \epsilon)$  and  $\mathcal{U}(t, \epsilon)$  such that

$$V(t, \epsilon) = t^{-(n-1)/2} \mathcal{V}(t, \epsilon), \quad U(t, \epsilon) = t^{-(n-1)/2} \left( \mathcal{U}(t, \epsilon) + \frac{n-1}{2\mu t} \mathcal{V}(t, \epsilon) \right), \quad (4.16)$$

where we use the notation

$$\mu = m + \omega, \quad \omega = \sqrt{m^2 - \epsilon^2}.$$

Below, we will omit the dependence of  $V$ ,  $U$ ,  $\mathcal{V}$ ,  $\mathcal{U}$ ,  $\omega$ , and  $\mu$  on  $\epsilon$ . By Lemma 4.4, for  $t \geq T_1$ , one has  $\mathcal{V}(t) > 0$  (since so is  $V(t)$ ). Then, applying inequalities from Lemma 4.4 to the relation

$$\mathcal{U}(t, \epsilon) = t^{(n-1)/2} \left( U(t, \epsilon) - \frac{n-1}{2\mu t} V(t, \epsilon) \right)$$

and using  $\omega > m/2$ ,  $t \geq T_1 \geq 2n$  (cf. (2.10) and (4.4)), we obtain:

$$\mathcal{U} \geq t^{(n-1)/2} \left( \frac{V}{4m} - \frac{n-1}{2(3m/2)2n} V \right) \geq t^{(n-1)/2} \frac{V}{12m} = \frac{\mathcal{V}}{12m} > 0, \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.17)$$

Substituting the expressions (4.16) into the system (3.13), we obtain the equation

$$\partial_t \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) - \frac{n-1}{2t} \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) + \frac{n-1}{t} \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) + \frac{\mathcal{V}}{\mu} = \epsilon^{-2} f \mathcal{V},$$

which takes the form

$$\partial_t \mathcal{U} + \frac{n-1}{2\mu t} \partial_t \mathcal{V} + \frac{n-1}{2t} \mathcal{U} + \frac{(n-1)^2 \mathcal{V}}{4\mu t^2} - \frac{(n-1) \mathcal{V}}{2\mu t^2} + \frac{\mathcal{V}}{\mu} = \frac{f}{\epsilon^2} \mathcal{V}, \quad (4.18)$$

and the equation

$$\partial_t \mathcal{V} - \frac{n-1}{2t} \mathcal{V} + \mu \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) = \partial_t \mathcal{V} + \mu \mathcal{U} = \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) f. \quad (4.19)$$

Above,  $f$  is evaluated at  $\tau = \epsilon^{2/k} V(t, \epsilon)^2 - \epsilon^{2+2/k} U(t, \epsilon)^2$ . Multiplying (4.18) by  $\mu$  and adding (4.19), we get:

$$\begin{aligned} \partial_t (\mathcal{V} + \mu \mathcal{U}) + (\mathcal{V} + \mu \mathcal{U}) + \frac{n-1}{2t} \partial_t \mathcal{V} + \frac{\mu(n-1)}{2t} \mathcal{U} + \frac{(n-1)(n-3)\mathcal{V}}{4t^2} \\ = \mu \frac{f}{\epsilon^2} \mathcal{V} + \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) f. \end{aligned} \quad (4.20)$$

Using (4.19) to simplify the two terms in the left-hand side which contain a factor  $(n-1)/(2t)$ , we get

$$\begin{aligned} \partial_t (\mathcal{V} + \mu \mathcal{U}) + (\mathcal{V} + \mu \mathcal{U}) + \frac{n-1}{2t} \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) f + \frac{(n-1)(n-3)\mathcal{V}}{4t^2} \\ = \mu \frac{f}{\epsilon^2} \mathcal{V} + \left( \mathcal{U} + \frac{n-1}{2\mu t} \mathcal{V} \right) f, \end{aligned}$$

which yields the inequality

$$|\partial_t (\mathcal{V} + \mu \mathcal{U}) + (\mathcal{V} + \mu \mathcal{U})| \leq \frac{C}{t^2} (\mathcal{V} + \mathcal{U}) + C \frac{|f|}{\epsilon^2} (\mathcal{V} + \mathcal{U}), \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1), \quad (4.21)$$

with some  $C < \infty$ ; we took into account that both  $\mathcal{V}$  and  $\mathcal{U}$  are positive (cf. (4.17)). Since one has  $0 < \frac{\mathcal{V}}{\mathcal{V} + \mu \mathcal{U}} \leq 1$  and  $0 < \frac{\mathcal{U}}{\mathcal{V} + \mu \mathcal{U}} \leq \frac{1}{\mu} \leq \frac{1}{m}$ , it follows from (4.21) that there is  $C' < \infty$  such that

$$-1 - \frac{c}{t^2} - C' \frac{|f|}{\epsilon^2} \leq \frac{\partial_t (\mathcal{V} + \mu \mathcal{U})}{\mathcal{V} + \mu \mathcal{U}} \leq -1 + \frac{c}{t^2} + C' \frac{|f|}{\epsilon^2}, \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.22)$$



We note that, by (3.4),

$$\frac{|f(\epsilon^{2/k}(V(t, \epsilon)^2 - \epsilon^2 U(t, \epsilon)^2))|}{\epsilon^2} \leq 2|V(t, \epsilon)^2 - \epsilon^2 U(t, \epsilon)^2|^k,$$

which is bounded and exponentially decreasing as  $t \rightarrow +\infty$  (uniformly in  $\epsilon \in (0, \epsilon_1)$ ) due to the exponential decay of  $V(t, \epsilon) = \hat{V}(t) + \tilde{V}(t, \epsilon)$ ,  $U(t, \epsilon) = \hat{U}(t) + \tilde{U}(t, \epsilon)$  in  $t$ , which we proved in Theorem 2.1. Thus,

$$\int_{T_1}^{\infty} \left( \frac{C'}{t^2} + C' \frac{|f|}{\epsilon^2} \right) dt \leq C''$$

is bounded by some  $C'' < \infty$  which does not depend on  $\epsilon \in (0, \epsilon_1)$ . This allows us to integrate (4.22) from  $T_1$  to an arbitrary value  $t \geq T_1$ ; we get

$$-(t - T_1) - C'' \leq \ln(\mathcal{V}(t) + \mu \mathcal{U}(t)) - \ln(\mathcal{V}(T_1) + \mu \mathcal{U}(T_1)) \leq -(t - T_1) + C'',$$

which yields the desired inequalities (4.13).  $\square$

The following result immediately follows from the inequality (4.14) in Lemma 4.5 due to  $\inf_{|t| \leq T_1} \hat{V} > 0$ .

**Corollary 4.6.** *There are  $C_1^* > c_1^* > 0$  such that*

$$\hat{V}(t) \geq c_1^* \langle t \rangle^{-(n-1)/2} e^{-|t|}, \quad \hat{V}(t) + |\hat{U}(t)| \leq C_1^* \langle t \rangle^{-(n-1)/2} e^{-|t|}, \quad \forall t \in \mathbb{R}.$$

We claim that the bound (4.15) from Lemma 4.5 could be improved as follows.

**Lemma 4.7.** *There is  $C_2 < \infty$  and  $T_2 \in (T_1, +\infty)$  such that*

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq C_2 h(\epsilon) t^{-(n-1)/2} e^{-t}, \quad \forall t \geq T_2, \quad \forall \epsilon \in (0, \epsilon_1),$$

with  $h(\epsilon)$  from (3.6).

Above,  $\epsilon_1 > 0$  is from Theorem 2.1 (2) and  $T_1 < \infty$  is as in Lemma 4.5.

*Proof.* We define  $\tilde{\mathcal{V}}(t, \epsilon)$ ,  $\tilde{\mathcal{U}}(t, \epsilon)$  by the relations similar to (4.16):

$$\tilde{V}(t, \epsilon) = t^{-\frac{n-1}{2}} \tilde{\mathcal{V}}(t, \epsilon), \quad \tilde{U}(t, \epsilon) = t^{-\frac{n-1}{2}} \left( \tilde{\mathcal{U}}(t) + \frac{n-1}{2\mu t} \tilde{\mathcal{V}}(t, \epsilon) \right), \quad (4.23)$$

where we denote

$$\mu = m + \omega, \quad \omega = \sqrt{m^2 - \epsilon^2}, \quad \epsilon \in (0, \epsilon_1).$$

By (3.14), the functions  $\tilde{\mathcal{V}}$ ,  $\tilde{\mathcal{U}}$  satisfy

$$\partial_t \left( \tilde{\mathcal{U}} + \frac{n-1}{2\mu t} \tilde{\mathcal{V}} \right) + \frac{n-1}{2t} \left( \tilde{\mathcal{U}} + \frac{n-1}{2\mu t} \tilde{\mathcal{V}} \right) + \frac{\tilde{\mathcal{V}}}{\mu} = (1+2k) \hat{V}^{2k} \tilde{\mathcal{V}} - t^{\frac{n-1}{2}} G_1$$

and

$$\partial_t \tilde{\mathcal{V}} - \frac{n-1}{2t} \tilde{\mathcal{V}} + \mu \left( \tilde{\mathcal{U}} + \frac{n-1}{2\mu t} \tilde{\mathcal{V}} \right) = t^{\frac{n-1}{2}} G_2,$$

which we rewrite as

$$\partial_t \tilde{\mathcal{U}} + \frac{n-1}{2\mu t} \partial_t \tilde{\mathcal{V}} + \frac{n-1}{2t} \tilde{\mathcal{U}} + \frac{(n-1)(n-3)}{4\mu t^2} \tilde{\mathcal{V}} + \frac{\tilde{\mathcal{V}}}{\mu} = (1+2k) \hat{V}^{2k} \tilde{\mathcal{V}} - t^{\frac{n-1}{2}} G_1, \quad (4.24)$$

$$\partial_t \tilde{\mathcal{V}} + \mu \tilde{\mathcal{U}} = t^{\frac{n-1}{2}} G_2. \quad (4.25)$$

We multiply (4.24) by  $\mu$ ; adding and subtracting (4.25), we obtain, respectively:

$$\begin{aligned}\partial_t(\mu\tilde{\mathcal{U}} + \tilde{\mathcal{V}}) + (\mu\tilde{\mathcal{U}} + \tilde{\mathcal{V}}) &= (1 + 2k)\mu\hat{V}^{2k}\tilde{\mathcal{V}} - \frac{(n-1)(n-3)}{4t^2}\tilde{\mathcal{V}} + t^{\frac{n-1}{2}}\left(G_2 - \mu G_1 - \frac{n-1}{2t}G_2\right) \\ \partial_t(\mu\tilde{\mathcal{U}} - \tilde{\mathcal{V}}) - (\mu\tilde{\mathcal{U}} - \tilde{\mathcal{V}}) &= (1 + 2k)\mu\hat{V}^{2k}\tilde{\mathcal{V}} - \frac{(n-1)(n-3)}{4t^2}\tilde{\mathcal{V}} - t^{\frac{n-1}{2}}\left(\mu G_1 + \frac{n-1}{2t}G_2 + G_2\right).\end{aligned}$$

Multiplying the above relations by  $e^t$  and  $e^{-t}$ , respectively, we rewrite them as

$$\partial_t(e^t(\mu\tilde{\mathcal{U}} + \tilde{\mathcal{V}})) = e^t\left((1 + 2k)\mu\hat{V}^{2k}\tilde{\mathcal{V}} - \frac{(n-1)(n-3)}{4t^2}\tilde{\mathcal{V}} + t^{\frac{n-1}{2}}\left(G_2 - \mu G_1 - \frac{n-1}{2t}G_2\right)\right), \quad (4.26)$$

$$\partial_t(e^{-t}(\mu\tilde{\mathcal{U}} - \tilde{\mathcal{V}})) = e^{-t}\left((1 + 2k)\mu\hat{V}^{2k}\tilde{\mathcal{V}} - \frac{(n-1)(n-3)}{4t^2}\tilde{\mathcal{V}} - t^{\frac{n-1}{2}}\left(\mu G_1 + \frac{n-1}{2t}G_2 + G_2\right)\right). \quad (4.27)$$

We are to integrate the above relations in  $t$ ; before we do this, we need a special treatment for the last term in the right-hand side of (4.26).

**Lemma 4.8.** *There is  $C < \infty$  such that*

$$\left| \int_T^{T'} t^{\frac{n-1}{2}} e^t \left( G_2 - \mu G_1 - \frac{n-1}{2t} G_2 \right) dt \right| \leq Ch(\epsilon), \quad \forall T' \geq T \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1),$$

with  $h(\epsilon)$  from (3.6).

*Proof.* Applying the bounds on  $G_1$  and  $G_2$  from Lemma 3.5, we can treat all the terms (obtaining the desired bound  $O(h(\epsilon))$ ) except for the ones linear in  $\hat{V}$  and  $\hat{U}$ ; the worry comes from e.g.  $\langle t \rangle^{(n-1)/2} e^t \hat{V}(t) \geq c_1^* > 0$  (cf. Corollary 4.6), whose contribution to the integral considered in the lemma would not be bounded uniformly in  $T, T'$ ; let us try to combine all such terms. The expression  $G_2 - \mu G_1 - \frac{n-1}{2t} G_2$  contributes the following terms which are linear in  $\hat{V}$  and  $\hat{U}$ :

$$(m - \omega)\hat{U} - (m + \omega)\frac{m - \omega}{2m(m + \omega)}\hat{V} - \frac{n-1}{2t}(m - \omega)\hat{U} = (m - \omega)\left(\hat{U} - \frac{\hat{V}}{2m} - \frac{n-1}{2t}\hat{U}\right).$$

Using (2.8), we rewrite the above as  $(m - \omega)\left(\hat{U} + \hat{U}' + \frac{n-1}{2t}\hat{U} - |\hat{V}|^{2k}\hat{V}\right)$ . Since

$$\int_T^{T'} t^{\frac{n-1}{2}} e^t \left( \hat{U} + \hat{U}' + \frac{n-1}{2t}\hat{U} - |\hat{V}|^{2k}\hat{V} \right) dt = \int_T^{T'} \partial_t(t^{\frac{n-1}{2}} e^t \hat{U}) dt - \int_T^{T'} t^{\frac{n-1}{2}} e^t |\hat{V}|^{2k} \hat{V} dt,$$

with both integrals in the right-hand side being bounded uniformly in  $T' \geq T \geq T_1$  (due to the bounds on  $\hat{U}$  and  $\hat{V}$  from Lemma 4.5), while  $m - \omega = O(\epsilon^2)$ , the conclusion follows.  $\square$

For some fixed  $T_2 \geq T_1$  (to be specified later), we denote

$$M(\epsilon) = \sup_{t \geq T_2} e^t \left( |\tilde{\mathcal{V}}(t, \epsilon)| + |\tilde{\mathcal{U}}(t, \epsilon)| \right), \quad \epsilon \in (0, \epsilon_1). \quad (4.28)$$

We note that, due to the bounds (4.15) from Lemma 4.5 and the definitions (4.23), one has

$$\sup_{\epsilon \in (0, \epsilon_1)} M(\epsilon) < \infty.$$

Integrating (4.26) from  $T_2$  to some  $t \geq T_2$  and using Lemma 4.8, one gets:

$$\begin{aligned}& \left| e^t |\mu\tilde{\mathcal{U}}(t, \epsilon) + \tilde{\mathcal{V}}(t, \epsilon)| - e^{T_2} |\mu\tilde{\mathcal{U}}(T_2, \epsilon) + \tilde{\mathcal{V}}(T_2, \epsilon)| \right| \\ & \leq C \int_{T_2}^t \left( \hat{V}(s)^{2k} + \frac{1}{s^2} \right) e^s \tilde{\mathcal{V}}(s, \epsilon) ds + Ch(\epsilon).\end{aligned} \quad (4.29)$$

Taking into account that, due to Theorem 2.1 and (4.23), one has

$$|\tilde{\mathcal{V}}(t, \epsilon)| + |\tilde{\mathcal{W}}(t, \epsilon)| = O(h(\epsilon)), \quad \forall t \geq T_1, \quad \forall \epsilon \in (0, \epsilon_1), \quad (4.30)$$

and using (4.28), we rewrite (4.29) as

$$e^t |\mu \tilde{\mathcal{W}}(t, \epsilon) + \tilde{\mathcal{V}}(t, \epsilon)| \leq M(\epsilon) C \int_{T_2}^t \left( \hat{V}(s)^{2k} + \frac{1}{s^2} \right) ds + Ch(\epsilon), \quad (4.31)$$

with some  $C < \infty$  (which does not depend on  $\epsilon \in (0, \epsilon_1)$ ,  $T_2 \geq T_1$ , and  $t \geq T_2$ ).

We now integrate (4.27) from  $t \geq T_2$  to  $+\infty$ . Due to the presence of the factor  $e^{-t}$  in the right-hand side, the last term does not need a special treatment such as in Lemma 4.8: the bounds on  $G_1$  and  $G_2$  from Lemma 3.5 together with the exponential decay of  $V, U, \hat{V}, \hat{U}$  from Lemma 4.5 are sufficient. The integration yields

$$e^{-t} |\mu \tilde{\mathcal{W}}(t, \epsilon) - \tilde{\mathcal{V}}(t, \epsilon)| \leq C \int_t^\infty \left( \hat{V}^{2k}(s) + \frac{1}{s^2} \right) e^{-s} \tilde{\mathcal{V}}(s, \epsilon) ds + Ch(\epsilon) e^{-2t},$$

again with some  $C < \infty$  which does not depend on  $\epsilon \in (0, \epsilon_1)$ ,  $T_2$ , and  $t$ . We took into account that in the left-hand side the boundary term at  $t = \infty$  disappears due to (4.30). Using (4.28), we rewrite the above relation as

$$e^t |\mu \tilde{\mathcal{W}}(t, \epsilon) - \tilde{\mathcal{V}}(t, \epsilon)| \leq M(\epsilon) C \int_t^\infty e^{2t-2s} \left( \hat{V}^{2k}(s) + \frac{1}{s^2} \right) ds + Ch(\epsilon). \quad (4.32)$$

Since

$$|\tilde{\mathcal{V}}| + |\tilde{\mathcal{W}}| \leq \frac{|\mu \tilde{\mathcal{W}} + \tilde{\mathcal{V}}| + |\mu \tilde{\mathcal{W}} - \tilde{\mathcal{V}}|}{2} + \frac{|\mu \tilde{\mathcal{W}} + \tilde{\mathcal{V}}| + |\mu \tilde{\mathcal{W}} - \tilde{\mathcal{V}}|}{2\mu},$$

the inequalities (4.31) and (4.32) lead to the bound

$$M(\epsilon) \leq M(\epsilon) C \int_{T_2}^\infty \left( \hat{V}(s)^{2k} + \frac{1}{s^2} \right) ds + Ch(\epsilon), \quad \forall \epsilon \in (0, \epsilon_1), \quad (4.33)$$

with some constant  $C < \infty$  (which does not depend on  $\epsilon \in (0, \epsilon_1)$  and  $T_2$ ). Now we can choose  $T_2$ : we set  $T_2 \geq T_1$  to be sufficiently large so that the coefficient at  $M(\epsilon)$  in the right-hand side is smaller than  $1/2$  (due to the exponential decay of  $\hat{V}$  and  $\tilde{\mathcal{V}}$ , such a value of  $T_2$  could be chosen independent of  $\epsilon \in (0, \epsilon_1)$ ). Now (4.33) turns into the inequality  $M(\epsilon) \leq 2Ch(\epsilon)$ , valid for all  $\forall \epsilon \in (0, \epsilon_1)$ , and (4.28) gives

$$|\tilde{\mathcal{V}}(t)| + |\tilde{\mathcal{W}}(t)| \leq 2Ch(\epsilon) e^{-t}, \quad \forall t \geq T_2, \quad \forall \epsilon \in (0, \epsilon_1),$$

yielding the bounds stated in the lemma.  $\square$

**Lemma 4.9.** *There is  $C_3 < \infty$  such that*

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq C_3 h(\epsilon) \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1), \quad (4.34)$$

with  $h(\epsilon)$  from (3.6).

Above,  $\epsilon_1 > 0$  is from Theorem 2.1 (2).

*Proof.* Using the bound from below on  $\hat{V}$  from Lemma 4.5 and bound from above on  $\tilde{V}$  and  $\tilde{U}$  from Lemma 4.7, we conclude that the inequality (4.34) takes place for  $t \geq T_2$  (and also for  $t \leq -T_2$ ) and for all  $\epsilon \in (0, \epsilon_1)$ . Let us now consider the case  $|t| \leq T_2$ . By the inequality (3.49), there is  $C < \infty$  such that

$$\|\tilde{V}(\cdot, \epsilon)\|_{L^\infty} + \|\tilde{U}(\cdot, \epsilon)\|_{L^\infty} \leq Ch(\epsilon) \leq \frac{C}{\hat{V}(T_2)} h(\epsilon) \hat{V}(t), \quad \forall |t| \leq T_2, \quad \forall \epsilon \in (0, \epsilon_0);$$

in the last inequality, we used the fact that  $\hat{V}(t)$  is positive and monotonically decreasing for  $t > 0$ . This proves the desired inequality for  $|t| \leq T_2$ .  $\square$

Lemma 4.9 proves (2.18).

The pointwise bound (2.19) follows from the inequality  $\hat{V}(t) \leq C_1^* \langle t \rangle^{-(n-1)/2} e^{-|t|}$  for  $t \in \mathbb{R}$  and  $\epsilon \in (0, \epsilon_1)$  (cf. Corollary 4.6) and also from (2.16) and (2.18) which show that  $\tilde{V}$ ,  $\hat{U}$ , and  $\tilde{U}$  are all pointwise dominated by  $\hat{V}$ .

This completes the proof of Theorem 2.1 (3).

For our convenience, we take  $\epsilon_1$  small enough so that  $C_3 h(\epsilon_1) < 1/2$ ; then, for the later use, we have

$$|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| < \frac{1}{2} \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.35)$$

## 5 Improved error estimates

Now we prove Theorem 2.1 (5). The assumption (2.21), together with the bounds on the amplitude of solitary waves (3.3), allows us to assume that there is  $c < \infty$  such that

$$|f(\tau) - |\tau|^k| \leq c|\tau|^K, \quad |f(\tau)| \leq (c+1)|\tau|^k, \quad \tau \in \mathbb{R}. \quad (5.1)$$

The improvement of the estimates stated in Theorem 2.1 (1) and (3) comes from having better bounds onto the second and third terms from the right-hand side of (3.37): when estimating e.g.  $|V^2 - \epsilon^2 U^2|^k - |V^2|^k$ , we no longer have to rely on Lemma 3.4, being able to use the Taylor expansions instead.

We recall that  $\Lambda_k < \infty$  was defined in (3.1).

**Lemma 5.1.** *There is  $C_4 < \infty$  such that for any numbers  $\hat{V}, \hat{U}, \tilde{V}, \tilde{U} \in [-\Lambda_k, \Lambda_k]$ ,  $V = \hat{V} + \tilde{V}$ , and  $U = \hat{U} + \tilde{U}$  which satisfy*

$$\epsilon_1 |U| \leq \frac{1}{2} V, \quad |\tilde{V}| \leq \frac{1}{2} \hat{V}, \quad (5.2)$$

one has

$$\left| f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 \hat{V}^{2k} \right| \leq C_4 \left( \epsilon^{2+2\kappa} \hat{V}^{2k} + \epsilon^2 \hat{V}^{2k-1} |\tilde{V}| \right), \quad \forall \epsilon \in (0, \epsilon_1).$$

Above,

$$\kappa := \min \left( 1, \frac{K}{k} - 1 \right)$$

was defined in (2.24).

*Proof.* We proceed as follows:

$$\begin{aligned} & \left| f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 \hat{V}^{2k} \right| \\ & \leq \left| f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 (V^2 - \epsilon^2 U^2)^k \right| + \epsilon^2 |(V^2 - \epsilon^2 U^2)^k - V^{2k}| + \epsilon^2 |V^{2k} - \hat{V}^{2k}| \\ & \leq c \epsilon^{2K/k} (V^2 - \epsilon^2 U^2)^K + O(\epsilon^2 V^{2(k-1)} \epsilon^2 U^2) + O(\epsilon^2 \hat{V}^{2k-1} \tilde{V}), \end{aligned}$$

where the three terms from the second line were estimated using (5.1) and (5.2). The conclusion follows.  $\square$

Here is an improvement of Lemma 3.5.

**Lemma 5.2.** *There is  $C < \infty$  such that for any numbers  $\hat{V}, \hat{U}, \tilde{V}, \tilde{U} \in [-\Lambda_k, \Lambda_k]$ ,  $V = \hat{V} + \tilde{V}$ , and  $U = \hat{U} + \tilde{U}$  which satisfy (5.2) and additionally*

$$|\hat{U}| \leq \frac{C_1}{c_1} \hat{V}, \quad (5.3)$$

with  $c_1$  and  $C_1$  from Lemma 4.5, one has:

$$\begin{aligned} \left| G_1 - \left( \frac{1}{m+\omega} - \frac{1}{2m} \right) \hat{V} - k\epsilon^2 V^{2k-1} U^2 \right| &\leq C((\epsilon^{2\frac{K}{k}-2} + \epsilon^4) \hat{V}^{2k+1} + \hat{V}^{2k-1} \tilde{V}^2), \\ |G_2 - \epsilon^2 \hat{V}^{2k} \hat{U} - (m-\omega) \hat{U}| &\leq C(\epsilon^{2+2\kappa} \hat{V}^{2k} + \epsilon^2 \hat{V}^{2k-1} \tilde{V}) |U| + \epsilon^2 \hat{V}^{2k} |\tilde{U}|, \\ |G_1| + |G_2| &\leq C\epsilon^{2\kappa} \hat{V} + C\hat{V}^{2k-1} \tilde{V}^2, \end{aligned}$$

for all  $\epsilon \in (0, \epsilon_1)$ . Above,  $G_1 = G_1(\epsilon, \tilde{V}, \tilde{U})$  and  $G_2 = G_2(\epsilon, \tilde{V}, \tilde{U})$ .

*Proof.* We start with the definition (3.15) of  $G_1(\epsilon, \tilde{V}, \tilde{U})$  and apply the inequalities (5.2):

$$\begin{aligned} G_1(\epsilon, \tilde{V}, \tilde{U}) &= -\epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) V + \hat{V}^{2k} \hat{V} + (1+2k) \hat{V}^{2k} \tilde{V} + \frac{\hat{V}}{m+\omega} - \frac{\hat{V}}{2m} \\ &= -(\epsilon^{-2} f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - |V^2 - \epsilon^2 U^2|^k) V - (|V^2 - \epsilon^2 U^2|^k - V^{2k}) V \\ &\quad - (V^{2k+1} - \hat{V}^{2k+1} - (1+2k) \hat{V}^{2k} \tilde{V}) + \left( \frac{1}{m+\omega} - \frac{1}{2m} \right) \hat{V} \\ &= O(\epsilon^{2\frac{K}{k}-2} V^{2K+1}) + k\epsilon^2 V^{2k-1} U^2 + O(\epsilon^4 V^{2k-3} U^4) + O(\hat{V}^{2k-1} \tilde{V}^2) + \left( \frac{1}{m+\omega} - \frac{1}{2m} \right) \hat{V}. \end{aligned}$$

Let us point out that the third term in the right hand side in the line above has the factor of  $\epsilon^4$ , which contributes  $\epsilon^4$  into the first conclusion of the lemma.

For  $G_2(\epsilon, \tilde{V}, \tilde{U})$  from (3.16), we have:

$$G_2 - \epsilon^2 \hat{V}^{2k} \hat{U} - (m-\omega) \hat{U} = f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) U - \epsilon^2 \hat{V}^{2k} \hat{U} = (f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)) - \epsilon^2 \hat{V}^{2k}) U + \epsilon^2 \hat{V}^{2k} \tilde{U}.$$

Applying Lemma 5.1 to the right-hand side, we have:

$$\left| G_2 - \epsilon^2 \hat{V}^{2k} \hat{U} - (m-\omega) \hat{U} \right| \leq C_4 \left( \epsilon^{2+2\kappa} \hat{V}^{2k} + \epsilon^2 \hat{V}^{2k-1} |\tilde{V}| \right) |U| + \epsilon^2 \hat{V}^{2k} |\tilde{U}|.$$

The second conclusion of the lemma follows.

Taking into account (5.2), the bound on  $|G_1| + |G_2|$  also follows; we need to mention that, due to (5.2) and (5.3), both  $|\hat{U}|$  and  $|\tilde{U}|$  are estimated by  $\hat{V}$ .  $\square$

We notice that, due to (2.16) and (4.35), the functions  $\hat{V}(t)$ ,  $\hat{U}(t)$ ,  $\tilde{V}(t, \epsilon)$ , and  $\tilde{U}(t, \epsilon)$  satisfy inequalities (5.2) for all  $t \in \mathbb{R}$  and  $\epsilon \in (0, \epsilon_1)$ . Also,  $\hat{U}(t)$  and  $\hat{V}(t)$  satisfy the inequality (5.3) due to (4.14) from Lemma 4.5. Using Lemma 5.2 in place of Lemma 3.5, we can rewrite the proof of Lemma 3.6 as follows.

**Lemma 5.3.** *There is  $C < \infty$  such that for any  $\epsilon \in (0, \epsilon_1)$  and any  $\begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \in X_{e,o}$  which satisfies*

$$\left\| e^{\gamma \langle t \rangle} \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\|_X \leq \left\| e^{\gamma \langle t \rangle} \begin{bmatrix} \hat{V}(t) \\ \hat{U}(t) \end{bmatrix} \right\|_X, \quad (5.4)$$

$$\epsilon_1 |U(t, \epsilon)| \leq \frac{1}{2} V(t, \epsilon), \quad |\tilde{V}(t, \epsilon)| \leq \frac{1}{2} \hat{V}(t, \epsilon), \quad \forall t \in \mathbb{R}, \quad \forall \epsilon \in (0, \epsilon_1), \quad (5.5)$$

where  $V(t, \epsilon) = \hat{V}(t) + \tilde{V}(t, \epsilon)$  and  $U(t, \epsilon) = \hat{U}(t) + \tilde{U}(t, \epsilon)$ , one has

$$\|e^{(1+2k)\gamma \langle t \rangle} G(\epsilon, \tilde{W})\|_X \leq C \left( \epsilon^{2\kappa} + \|e^{\gamma \langle t \rangle} \tilde{W}\|_X^2 \right), \quad (5.6)$$

with  $\kappa$  from (2.24).

*Proof.* For  $\tilde{V}(t, \epsilon)$  and  $\tilde{U}(t, \epsilon)$  as in the assumptions of the lemma, due to Lemma 5.2, one has

$$|G_1(\epsilon, \tilde{V}, \tilde{U})| + |G_2(\epsilon, \tilde{V}, \tilde{U})| \leq C\epsilon^{2\kappa}\hat{V}(t, \epsilon) + C\hat{V}(t, \epsilon)^{2k-1}\tilde{V}(t, \epsilon)^2, \quad \forall \epsilon \in (0, \epsilon_1), \quad \forall t \in \mathbb{R}. \quad (5.7)$$

Multiplying the first term in the right-hand side by  $e^{(1+2k)\gamma\langle t \rangle}$  and using (3.39) and (5.4), we bound the resulting  $X$ -norm by  $C\epsilon^{2\kappa}$ , with some  $C < \infty$ . The second term in the right-hand side of (5.7) is homogeneous of order  $1 + 2k$  in  $\tilde{V}$  and  $\hat{V}$ ; we multiply it by the factor  $e^{(1+2k)\gamma\langle t \rangle}$ , absorbing  $e^{\gamma\langle t \rangle}$  into each power of  $\hat{V}$  and  $\tilde{V}$  and bounding the  $X$ -norm of the result by  $C\|e^{\gamma\langle t \rangle}\hat{V}\|_X^{2k-1}\|e^{\gamma\langle t \rangle}\tilde{V}\|_X^2 \leq C'\|e^{\gamma\langle t \rangle}\tilde{V}\|_X^2$ .  $\square$

Now we use Lemma 5.3 to improve the estimates on  $\tilde{W}$ .

**Lemma 5.4.** *One can take  $\epsilon_1 > 0$  smaller if necessary so that, for some  $b_1 > 0$ ,*

$$\|e^{\gamma\langle t \rangle}\tilde{W}(\epsilon)\|_{X^1} \leq b_1\epsilon^{2\kappa}, \quad \forall \epsilon \in (0, \epsilon_1).$$

*Proof.* We recall the relation (3.42) satisfied by  $\tilde{W}$ :

$$e^{\gamma\langle t \rangle}\tilde{W} = e^{-2k\gamma\langle t \rangle}A_\gamma(\epsilon)^{-1}e^{(1+2k)\gamma\langle t \rangle}G(\epsilon, e^{-\gamma\langle t \rangle}e^{\gamma\langle t \rangle}\tilde{W}).$$

Using the continuity of the mapping (3.41) and estimating  $G(\epsilon, \tilde{W})$  with the aid of Lemma 5.3, we obtain:

$$\|e^{\gamma\langle t \rangle}\tilde{W}\|_{X^1} = \|e^{-2k\gamma\langle t \rangle}A_\gamma(\epsilon)^{-1}\|_{X \rightarrow X^1}\|e^{(1+2k)\gamma\langle t \rangle}G(\epsilon, \tilde{W})\|_X \leq C(\epsilon^{2\kappa} + \|e^{\gamma\langle t \rangle}\tilde{W}\|_X^2).$$

Since  $\|e^{\gamma\langle t \rangle}\tilde{W}\|_X \leq \|e^{\gamma\langle t \rangle}\tilde{W}\|_{X^1}$  (cf. (1.7)), the above relation yields the bound stated in the lemma as long as  $\|e^{\gamma\langle t \rangle}\tilde{W}\|_{X^1}$  is sufficiently small (which holds due to (3.49) as long as  $\epsilon \in (0, \epsilon_1)$  with  $\epsilon_1 > 0$  small enough).  $\square$

Lemma 5.4 improves the estimates on the error terms  $\tilde{V}, \tilde{U}$  formulated in Theorem 2.1 (I), proving (2.22).

We also do the second pass over the proof of Theorem 2.1 (3), improving in (2.18) the factor  $h(\epsilon)$  to  $\epsilon^{2\kappa}$ . For this, we rewrite the proof of Lemma 4.7, where the bounds on  $G_1, G_2$  come from Lemma 5.2 instead of Lemma 3.5. We also rewrite the proof of Lemma 4.9 with  $\epsilon^{2\kappa}$  instead of  $h(\epsilon)$  (we use (2.22) in place of (3.49)). This brings us at  $|\tilde{V}(t, \epsilon)| + |\tilde{U}(t, \epsilon)| \leq C\epsilon^{2\kappa}\hat{V}(t)$ , with some  $C < \infty$ , valid for all  $t \in \mathbb{R}$  and all  $\epsilon \in (0, \epsilon_1)$ , thus proving (2.23).

This completes the proof of Theorem 2.1.

## 6 Solitary waves in the nonrelativistic limit. The case $f \in C^1$

We now turn to the case when  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$  satisfies both the assumption (2.25) and (2.26). Just like the former assumption leads to (5.1), the assumption (2.26) allows us to accept that there is  $C < \infty$  such that

$$|\tau f'(\tau) - k|\tau|^k| \leq C|\tau|^K, \quad |\tau f'(\tau)| \leq (C + k)|\tau|^k, \quad \tau \in \mathbb{R}, \quad (6.1)$$

where  $k \in (0, 2/(n-2))$  (any  $k > 0$  if  $n \leq 2$ ) and  $K > k$ . Now we will be able to prove uniqueness and regularity of the family of solitary waves bifurcating from the nonrelativistic limit. This amounts to noticing that in (3.45), taking into account Theorem 2.1 (2), we actually recover some features of the implicit function theorem. A careful analysis shows that the main obstacle to its application is the lack of regularity of the mapping  $f$  in (3.15), (3.16). This closer look shows that the unique obstacle are the terms  $f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2))V$  and  $f(\epsilon^{2/k}(V^2 - \epsilon^2 U^2))U$ , which with (6.1) can now be treated.

## 6.1 Improved regularity of the groundstate

Let us prove Theorem 2.2 (I). By Theorem 2.1, we already have  $\phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$ ,  $\omega \in (\omega_1, m)$ , with  $\omega_1 = \sqrt{m^2 - \epsilon_1^2}$ , with  $\epsilon_1 > 0$  from Theorem 2.1 (2); we need to show how to get the improvement in the regularity of  $\phi_\omega$  under better regularity of  $f$ .

We start with the improvement of regularity of  $V, U$  proved in Lemma 3.9.

**Lemma 6.1.** *Let  $\omega \in (\omega_1, m)$ . If in (3.13) one has  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$  which satisfies (6.1), and if  $V, U \in C^1(\mathbb{R})$ , with  $V$  even and  $U$  odd, then  $V, U \in C^2(\mathbb{R})$ , and  $H(t) = U(t)/t$  could be extended to a function  $H \in C^1(\mathbb{R})$ .*

*Proof.* First we consider the case  $f \in C^1(\mathbb{R})$ . We proceed similarly to Lemma 3.9. The inclusion  $V \in C^2(\mathbb{R})$  immediately follows from the second equation in (3.13). Let us prove that  $U \in C^2(\mathbb{R})$ . Equation (3.13) takes the form (3.50) with

$$B(t) = \frac{f}{\epsilon^2} V(t) - \frac{1}{m + \omega} V(t), \quad f = f(\epsilon^{\frac{2}{k}} V(t)^2 - \epsilon^{2+\frac{2}{k}} U(t)^2).$$

We note that now  $B \in C^1(\mathbb{R})$  and is even. It is enough to prove that  $H(t) = U(t)/t$  could be extended to a  $C^1$  function on  $\mathbb{R}$ . Since  $H(t)$  is even, it is enough to prove that  $\lim_{t \rightarrow 0} H'(t) = 0$ . Taking the derivative of (3.52) at  $t > 0$ , we arrive at

$$H'(t) = \frac{B(t)t^{n-1}}{t^n} - n \frac{\int_0^t B(\tau)\tau^{n-1} d\tau}{t^{n+1}} = \frac{B(t)t^n - n \int_0^t B(\tau)\tau^{n-1} d\tau}{t^{n+1}} = \frac{\int_0^t B'(\tau)\tau^n d\tau}{t^{n+1}},$$

therefore

$$\lim_{t \rightarrow 0} H'(t) = \lim_{t \rightarrow 0} \frac{\int_0^t B'(\tau)\tau^n d\tau}{t^{n+1}} = \lim_{t \rightarrow 0} \frac{B'(t)}{n+1} = \frac{B'(0)}{n+1} = 0,$$

where we took into account that  $B \in C^1(\mathbb{R})$  is even.

The above argument still applies if we only require that  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ : due to Theorem 2.1, the argument of  $f$ , given by  $\tau(t) = \epsilon^{\frac{2}{k}} V(t)^2 - \epsilon^{2+\frac{2}{k}} U(t)^2$ , always belongs to  $\mathbb{R}_+ = (0, +\infty)$ , hence in (3.13) one has  $f(\tau(t))$  which is a  $C^1$  function of  $t \in \mathbb{R}$ . Moreover, one can deduce from (6.1) that  $\epsilon^{-2} V(t) \frac{d}{dt} f(\tau(t))$  remains bounded pointwise by  $C \hat{V}(t)^k (|V'(t)| + |U'(t)|)$  uniformly in  $t \in \mathbb{R}$  and in  $\epsilon \in (0, \epsilon_1)$ :

$$\begin{aligned} \left| \epsilon^{-2} V(t) \frac{d}{dt} f(\tau(t)) \right| &= |\epsilon^{-2} V(t) f'(\tau(t)) (2\epsilon^{\frac{2}{k}} V(t) V'(t) - 2\epsilon^{2+\frac{2}{k}} U(t) U'(t))| \\ &\leq \epsilon^{-2} |V(t)| \frac{k|\tau|^k + C|\tau|^K}{|\tau|} |2\epsilon^{\frac{2}{k}} V(t) V'(t) - 2\epsilon^{2+\frac{2}{k}} U(t) U'(t)| \\ &\leq \frac{C}{\epsilon^2} (k|\tau|^k + C|\tau|^K) (|V'| + |U'|) \leq C \hat{V}(t)^k (|V'(t)| + |U'(t)|). \end{aligned}$$

Above, we used (6.1) to deal with  $f'$  (note that  $\tau > 0$  by Theorem 2.1 (2) and (3)), and then Theorem 2.1 (3) to estimate  $|V(t)|$  and  $|U(t)|$  with the aid of  $\hat{V}(t)$ . So, we again have  $B \in C^1(\mathbb{R})$  and proceed as in the first part of the argument.  $\square$

Now we can show that  $\phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^N)$  for  $\omega \in (\omega_1, m)$ . From the Ansatz (2.11),

$$\phi_\omega(x) = \begin{bmatrix} v(r, \omega) \mathbf{n} \\ i u(r, \omega) (\mathbf{e}_r \cdot \boldsymbol{\sigma}) \mathbf{n} \end{bmatrix} = \begin{bmatrix} v(r, \omega) \mathbf{n} \\ i \frac{u(r, \omega)}{r} (x \cdot \boldsymbol{\sigma}) \mathbf{n} \end{bmatrix},$$

taking into account that  $H(t) = U(t)/t$  belongs to  $C^1(\mathbb{R})$  (as we proved in Lemma 6.1), we conclude that  $\phi_\omega \in C^1(\mathbb{R}^n, \mathbb{C}^N)$ . Therefore, the nonlinear term  $f(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega$  is in  $C^1(\mathbb{R}^n, \mathbb{C}^N)$  as a function of  $x \in \mathbb{R}^n$ , and one has:

$$\begin{aligned} |\nabla (f(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega)| &\leq |f'(\phi_\omega^* \beta \phi_\omega)| |\operatorname{Re}(\phi_\omega^* \beta \nabla \phi_\omega)| |\phi_\omega| + |f(\phi_\omega^* \beta \phi_\omega)| |\nabla \phi_\omega| \\ &\leq C (|f'(\phi_\omega^* \beta \phi_\omega)| |\phi_\omega|^2 + |f(\phi_\omega^* \beta \phi_\omega)|) |\nabla \phi_\omega|. \end{aligned} \quad (6.2)$$



By Theorem 2.1,  $\phi_\omega \in L^\infty(\mathbb{R}^n, \mathbb{C}^N)$  and  $\nabla \phi_\omega \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ ; using the bounds (5.1), (6.1), we conclude that the right-hand side of (6.2) is in  $L^2(\mathbb{R}^n)$ . Then (6.2) shows that  $f(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega$  is in  $H^1(\mathbb{R}^n, \mathbb{C}^N)$ , and then from

$$\phi_\omega = -(D_m - \omega)^{-1} f(\phi_\omega^* \beta \phi_\omega) \beta \phi_\omega,$$

with some  $\omega \in (\omega_1, m)$ , we deduce the inclusion  $\phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^N)$ .

## 6.2 Uniqueness, continuity, and differentiability of the mapping $\omega \mapsto \phi_\omega$

We start with the following technical result. Recall that  $\Lambda_k < \infty$  was defined in (3.1).

**Lemma 6.2.** *There is  $C < \infty$  such that for all  $\epsilon \in (0, \epsilon_1)$  and for all numbers*

$$\hat{V}, \hat{U}, \tilde{V}, \tilde{U} \in [-\Lambda_k, \Lambda_k], \quad V = \hat{V} + \tilde{V}, \quad U = \hat{U} + \tilde{U}$$

which satisfy

$$\epsilon_1 |U| \leq \frac{1}{2} V, \quad |\tilde{V}| \leq b_2 \epsilon^{2\kappa} \hat{V}, \quad (6.3)$$

one has

$$\left\| \frac{\partial G(\epsilon, \tilde{V}, \tilde{U})}{\partial(\tilde{V}, \tilde{U})} \right\|_{\text{End}(\mathbb{C}^2)} \leq C \epsilon^{2\kappa},$$

where  $G(\epsilon, \tilde{V}, \tilde{U}) = \begin{bmatrix} G_1(\epsilon, \tilde{V}, \tilde{U}) \\ G_2(\epsilon, \tilde{V}, \tilde{U}) \end{bmatrix}$  (cf. (3.15), (3.16)).

Above,  $\epsilon_1 > 0$  is from Theorem 2.1 (2) and  $b_2 < \infty$  is from Theorem 2.1 (5).

*Proof.* Denote  $V = \hat{V} + \tilde{V}$  and  $U = \hat{U} + \tilde{U}$ . Let us consider

$$\frac{\partial G(\epsilon, \tilde{V}, \tilde{U})}{\partial(\tilde{V}, \tilde{U})} = \begin{bmatrix} \partial_{\tilde{V}} G_1 & \partial_{\tilde{U}} G_1 \\ \partial_{\tilde{V}} G_2 & \partial_{\tilde{U}} G_2 \end{bmatrix} = \begin{bmatrix} -2f' \epsilon^{\frac{2}{k}-2} V^2 - \epsilon^{-2} f + (1+2k) \hat{V}^{2k} & 2f' \epsilon^{\frac{2}{k}} V U \\ 2f' \epsilon^{\frac{2}{k}} V U & f - 2f' \epsilon^{2+\frac{2}{k}} U^2 \end{bmatrix}.$$

Above,  $f$  and  $f'$  are evaluated at  $\tau = \epsilon^{2/k}(V^2 - \epsilon^2 U^2)$ . All the terms except for  $\partial_{\tilde{V}} G_1$  are immediately  $O(\epsilon^2)$ ; we now focus on  $\partial_{\tilde{V}} G_1$ . Denoting  $\tau = \epsilon^{2/k}(V^2 - \epsilon^2 U^2) = O(\epsilon^{2/k})$ , one has:

$$|\epsilon^{-2} f(\tau) - \hat{V}^{2k}| \leq |\epsilon^{-2} f(\tau) - \epsilon^{-2} \tau^k| + |(V^2 - \epsilon^2 U^2)^k - V^{2k}| + |V^{2k} - \hat{V}^{2k}| \leq C \epsilon^{2\kappa}.$$

We estimated the three terms in the middle using (5.1) and (6.3). Similarly,

$$|f'(\tau) \epsilon^{\frac{2}{k}-2} 2V^2 - 2k \hat{V}^{2k}| \leq \frac{2V^2 \epsilon^{\frac{2}{k}}}{\epsilon^2} |f'(\tau) - k \tau^{k-1}| + \frac{2k V^2 \epsilon^{\frac{2}{k}}}{\epsilon^2} |\tau^{k-1} - (\epsilon^{\frac{2}{k}} V^2)^{k-1}| + 2k |V^{2k} - \hat{V}^{2k}| \leq C \epsilon^{2\kappa};$$

we used (6.1) and (6.3). So,

$$|\partial_{\tilde{V}} G_1| = |-2f' \epsilon^{\frac{2}{k}-2} V^2 - \epsilon^{-2} f + (1+2k) \hat{V}^{2k}| \leq |\epsilon^{-2} f - \hat{V}^{2k}| + |2f' \epsilon^{\frac{2}{k}-2} V^2 - 2k \hat{V}^{2k}| = O(\epsilon^{2\kappa}). \quad \square$$

We claim that the mapping

$$\mu : X_{e,o} \rightarrow X_{e,o}^1 \subset X_{e,o}, \quad \mu : \tilde{W} \mapsto A(\epsilon)^{-1} G(\epsilon, \tilde{W})$$

is a contraction when considered on a certain subset of a sufficiently small ball.

**Lemma 6.3.** *Let  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$  satisfy (6.1). Then there is  $\epsilon_2 \in (0, \epsilon_1)$  such that for any  $\epsilon \in (0, \epsilon_2)$  and any*

$$\tilde{W}_0 = \begin{bmatrix} \tilde{V}_0 \\ \tilde{U}_0 \end{bmatrix} \in \overline{\mathbb{B}_\rho(X_{e,o})}, \quad \tilde{W}_1 = \begin{bmatrix} \tilde{V}_1 \\ \tilde{U}_1 \end{bmatrix} \in \overline{\mathbb{B}_\rho(X_{e,o})}, \quad \text{with } \rho = b_1 \epsilon^{2\kappa},$$

with  $b_1 > 0$  from Lemma 5.4, which satisfy

$$\epsilon_1 |\hat{U}(t) + \tilde{U}_s(t)| \leq \frac{1}{2} (\hat{V}(t) + \tilde{V}_s(t)), \quad \forall t \in \mathbb{R}, \quad \forall s = 0, 1, \quad (6.4)$$

$$|\tilde{V}_s(t)| \leq b_2 \epsilon_2^{2\kappa} \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall s = 0, 1, \quad (6.5)$$

one has

$$\|\mu(\epsilon, \tilde{W}_1) - \mu(\epsilon, \tilde{W}_0)\|_{X^1} \leq \frac{1}{2} \|\tilde{W}_1 - \tilde{W}_0\|_X.$$

Above,  $b_2 < \infty$  is from Lemma 6.2. We point out that, by Theorem 2.1 (2), the fixed points of  $\mu(\epsilon, \cdot)$  satisfy (6.4), and by Theorem 2.1 (3) these points also satisfy (6.5).

*Proof.* We consider the linear interpolations

$$\tilde{V}_s(t) = (1-s)\tilde{V}_0(t) + s\tilde{V}_1(t), \quad \tilde{U}_s(t) = (1-s)\tilde{U}_0(t) + s\tilde{U}_1(t), \quad s \in [0, 1],$$

and we also set

$$V_s(t) = \hat{V}(t) + \tilde{V}_s(t), \quad U_s(t) = \hat{U}(t) + \tilde{U}_s(t).$$

We notice that, due to (6.4) and (6.5), these interpolations are such that  $\begin{bmatrix} V_s \\ U_s \end{bmatrix} \in X_{e,o}$ ,  $\forall s \in [0, 1]$ , and they also satisfy the equivalents of (6.4) and (6.5) for all  $s \in [0, 1]$ ;

$$\begin{aligned} \epsilon_1 |U_s(t)| &\leq \frac{1}{2} V_s(t), \quad \forall t \in \mathbb{R}, \quad \forall s \in [0, 1], \\ |\tilde{V}_s(t)| &\leq b_2 \epsilon_2^{2\kappa} \hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall s \in [0, 1]. \end{aligned}$$

Let us pick  $\epsilon \in (0, \epsilon_1)$  and consider the relation

$$\mu(\epsilon, \tilde{W}_1) - \mu(\epsilon, \tilde{W}_0) = A(\epsilon)^{-1} (G(\epsilon, \tilde{W}_1) - G(\epsilon, \tilde{W}_0)). \quad (6.6)$$

To estimate the right-hand side, we consider

$$\begin{aligned} &G_1(\epsilon, \tilde{W}_1) - G_1(\epsilon, \tilde{W}_0) \\ &= -\epsilon^{-2} \left( f(\epsilon^{\frac{2}{k}}(V_1^2 - \epsilon^2 U_1^2)) V_1 - f(\epsilon^{\frac{2}{k}}(V_0^2 - \epsilon^2 U_0^2)) V_0 \right) + (1 + 2k) \hat{V}^{2k} (\tilde{V}_1 - \tilde{V}_0), \\ &G_2(\epsilon, \tilde{W}_1) - G_2(\epsilon, \tilde{W}_0) = f(\epsilon^{\frac{2}{k}}(V_1^2 - \epsilon^2 U_1^2)) U_1 - f(\epsilon^{\frac{2}{k}}(V_0^2 - \epsilon^2 U_0^2)) U_0. \end{aligned} \quad (6.7)$$

For (6.7), we have:

$$\begin{aligned} G(\epsilon, \tilde{W}_1) - G(\epsilon, \tilde{W}_0) &= \int_0^1 ds \frac{d}{ds} G(\epsilon, (1-s)\tilde{W}_0 + s\tilde{W}_1) \\ &= (\tilde{W}_1 - \tilde{W}_0) \int_0^1 \partial_{\tilde{W}} G(\epsilon, (1-s)\tilde{W}_0 + s\tilde{W}_1) ds. \end{aligned} \quad (6.8)$$

Applying Lemma 6.2 to (6.8), we have:

$$\|G(\epsilon, \tilde{W}_1) - G(\epsilon, \tilde{W}_0)\|_X \leq C \epsilon^{2\kappa} \|\tilde{W}_1 - \tilde{W}_0\|_X,$$

with some  $C < \infty$ . We take  $\epsilon_2 \in (0, \epsilon_1)$  so small that

$$C \epsilon_2^{2\kappa} \sup_{\epsilon \in [0, \epsilon_1]} \|A(\epsilon)^{-1}\|_{X_{e,o} \rightarrow X_{e,o}^1} \leq 1/2; \quad (6.9)$$

then the lemma follows from applying (6.9) to (6.6).  $\square$

For each  $\epsilon \in (0, \epsilon_2)$ , Lemma 6.3 proves the uniqueness of the fixed point of  $\mu(\epsilon, \cdot)$  in  $X_{e,o}$  which satisfies

$$\tilde{W} = A(\epsilon)^{-1}G(\epsilon, \tilde{W}), \quad \tilde{W} \in \overline{\mathbb{B}_\rho(X_{e,o})}, \quad \text{where } \rho = b_1\epsilon^{2\kappa};$$

this is the fixed point  $\tilde{W}$  which we constructed in Theorem 2.1. Thus, we have a well-defined map

$$(0, \epsilon_2) \rightarrow \mathbb{B}_\rho(X_{e,o}^1), \quad \rho = b_1\epsilon_2^{2\kappa}; \quad \epsilon \mapsto \tilde{W}(t, \epsilon), \quad \|e^{\gamma(t)}\tilde{W}(\cdot, \epsilon)\|_{H^1(\mathbb{R}, \mathbb{C}^2)} \leq b_1\epsilon^{2\kappa}. \quad (6.10)$$

The above argument also implies the continuity of the fixed point  $\tilde{W}(\epsilon)$  as a function of  $\epsilon$ , since for any  $\epsilon, \epsilon' \in (0, \epsilon_2)$  one has

$$\tilde{W}(\epsilon') - \tilde{W}(\epsilon) = A(\epsilon')^{-1}(G(\epsilon', \tilde{W}(\epsilon')) - G(\epsilon', \tilde{W}(\epsilon))) + A(\epsilon')^{-1}G(\epsilon', \tilde{W}(\epsilon)) - A(\epsilon)^{-1}G(\epsilon, \tilde{W}(\epsilon)).$$

We evaluate  $X$ -norm of the above relation, applying Lemma 6.3 to the first term in the right-hand side; this yields

$$\|\tilde{W}(\epsilon') - \tilde{W}(\epsilon)\|_X \leq 2\|A(\epsilon')^{-1}G(\epsilon', \tilde{W}(\epsilon)) - A(\epsilon)^{-1}G(\epsilon, \tilde{W}(\epsilon))\|_X, \quad \forall \epsilon, \epsilon' \in (0, \epsilon_2).$$

Due to the continuous dependence of  $A$  and  $G$  on  $\epsilon > 0$ , the above relation proves the continuity of the map (6.10) in  $\epsilon \in (0, \epsilon_2)$ .

We now turn to the differentiability of  $\tilde{W}$  with respect to  $\epsilon$ . Let us take  $\alpha, \beta \in (0, \epsilon_2)$  (with  $\epsilon_2 > 0$  from Lemma 6.3). Without loss of generality, we may assume that  $\alpha < \beta$ . For both  $\alpha$  and  $\beta$ , we denote the unique fixed points of  $\mu(\alpha, \cdot)$  and  $\mu(\beta, \cdot)$  (the images of  $\alpha, \beta \in (0, \epsilon_2)$  under the mapping (6.10)) by  $\tilde{W}(t, \alpha) = \begin{bmatrix} \tilde{V}(t, \alpha) \\ \tilde{U}(t, \alpha) \end{bmatrix}$  and  $\tilde{W}(t, \beta) = \begin{bmatrix} \tilde{V}(t, \beta) \\ \tilde{U}(t, \beta) \end{bmatrix}$ . By Theorem 2.1 (2) and (3), these fixed points satisfy

$$\begin{aligned} \epsilon_1|U(t, \alpha)| &\leq \frac{1}{2}V(t, \alpha), & |\tilde{V}(t, \alpha)| &\leq b_2\alpha^{2\kappa}\hat{V}(t), & \forall t \in \mathbb{R}, \\ \epsilon_1|U(t, \beta)| &\leq \frac{1}{2}V(t, \beta), & |\tilde{V}(t, \beta)| &\leq b_2\beta^{2\kappa}\hat{V}(t), & \forall t \in \mathbb{R}, \end{aligned}$$

therefore the linear interpolation

$$\tilde{W}_s(t) = \begin{bmatrix} \tilde{V}_s(t) \\ \tilde{U}_s(t) \end{bmatrix} = (1-s)\tilde{W}(t, \alpha) + s\tilde{W}(t, \beta), \quad \begin{bmatrix} V_s(t) \\ U_s(t) \end{bmatrix} = \begin{bmatrix} \hat{V}(t) + \tilde{V}_s(t) \\ \hat{U}(t) + \tilde{U}_s(t) \end{bmatrix}, \quad s \in [0, 1],$$

satisfies

$$\epsilon_1|U_s(t)| \leq \frac{1}{2}V_s(t), \quad |\tilde{V}_s(t)| \leq b_2\beta^{2\kappa}\hat{V}(t), \quad \forall t \in \mathbb{R}, \quad \forall s \in [0, 1]; \quad (6.11)$$

in the last inequality, we took into account that  $\alpha < \beta$ . We have:

$$\begin{aligned} \frac{\tilde{W}(\beta) - \tilde{W}(\alpha)}{\beta - \alpha} &= \frac{\mu(\beta, \tilde{W}(\beta)) - \mu(\beta, \tilde{W}(\alpha))}{\beta - \alpha} + \frac{\mu(\beta, \tilde{W}(\alpha)) - \mu(\alpha, \tilde{W}(\alpha))}{\beta - \alpha} \\ &= A(\beta)^{-1} \int_0^1 \partial_{\tilde{W}} G(\beta, (1-s)\tilde{W}(\alpha) + s\tilde{W}(\beta)) ds \frac{\tilde{W}(\beta) - \tilde{W}(\alpha)}{\beta - \alpha} + \frac{\mu(\beta, \tilde{W}(\alpha)) - \mu(\alpha, \tilde{W}(\alpha))}{\beta - \alpha}. \end{aligned}$$

The above relation takes place at each  $t \in \mathbb{R}$ ; we omitted the dependence on  $t$ . By Lemma 6.2, which is applicable due to (6.11), we can choose  $\epsilon_2 \in (0, \epsilon_1)$  smaller if necessary so that the operator

$$B(t, \alpha, \beta) := I_{\mathbb{C}^2} - A(\beta)^{-1} \int_0^1 \partial_{\tilde{W}} G(\beta, (1-s)\tilde{W}(t, \alpha) + s\tilde{W}(t, \beta)) ds, \quad B(t, \alpha, \beta) \in \text{End}(\mathbb{C}^2),$$

is invertible, with the inverse bounded uniformly in  $t \in \mathbb{R}$  and  $\alpha, \beta \in (0, \epsilon_2)$ ; we then have:

$$\frac{\tilde{W}(\beta) - \tilde{W}(\alpha)}{\beta - \alpha} = B(\alpha, \beta)^{-1} \frac{\mu(\beta, \tilde{W}(\alpha)) - \mu(\alpha, \tilde{W}(\alpha))}{\beta - \alpha}.$$

Since  $B$  is continuous in  $\alpha$  and  $\beta$  while  $\mu(\epsilon, \tilde{W}) = A(\epsilon)^{-1}G(\epsilon, \tilde{W})$ , with both  $A(\epsilon)^{-1}$  and  $G(\epsilon, \tilde{W})$  differentiable in  $\epsilon$ , we deduce that  $(\tilde{W}(t, \beta) - \tilde{W}(t, \alpha))/(\beta - \alpha)$  has a limit as  $\beta \rightarrow \alpha$ ; setting  $\alpha = \epsilon$ , we have:

$$\begin{aligned} \partial_\epsilon \tilde{W} &= B^{-1} \frac{\partial}{\partial \epsilon} (A^{-1}G(\epsilon, \tilde{W})) = B^{-1}A^{-1}(-\partial_\epsilon A A^{-1}G(\epsilon, \tilde{W}) + \partial_\epsilon G(\epsilon, \tilde{W})) \\ &= B^{-1}A^{-1}(-\partial_\epsilon A \tilde{W} + \partial_\epsilon G(\epsilon, \tilde{W})), \end{aligned} \quad (6.12)$$

where  $\tilde{W} = \tilde{W}(t, \epsilon)$ ,

$$A = A(\epsilon), \quad B = B(t, \epsilon) := B(t, \epsilon, \epsilon) = I_{\mathbb{C}^2} - A(\epsilon)^{-1} \partial_{\tilde{W}} G(\epsilon, \tilde{W}(t, \epsilon)). \quad (6.13)$$

In the last equality in (6.12), we took into account that  $\tilde{W}(t, \epsilon) = A(\epsilon)^{-1}G(\epsilon, \tilde{W})$  (cf. (3.19)).

**Lemma 6.4.** *One has:*

$$\begin{aligned} \left\| e^{\gamma(t)} \frac{\partial G}{\partial \epsilon}(\epsilon, \tilde{W}(t, \epsilon)) \right\|_X &= O(\epsilon^{2\kappa-1}), \quad \epsilon \in (0, \epsilon_2), \\ \left\| e^{\gamma(t)} \left( \frac{\partial G}{\partial \epsilon}(\epsilon, \tilde{W}(t, \epsilon)) - \epsilon \left[ \frac{2k\hat{U}^2\hat{V}^{2k-1} + \frac{\hat{V}}{4m^3}}{2\hat{U}\hat{V}^{2k} + \frac{\hat{U}}{m}} \right] \right) \right\|_X &= O(\epsilon^{\frac{2K}{k}-3}) + o(\epsilon), \quad \epsilon \in (0, \epsilon_2). \end{aligned}$$

*Proof.* Since  $2\kappa - 1 \leq 1$  and due to the exponential decay of  $\hat{V}$  and  $\hat{U}$  (cf. Lemma A.1), the first estimate stated in the lemma follows from the second one. By (3.15) and (3.16),  $\partial_\epsilon G$  is given by

$$\frac{\partial G(\epsilon, \tilde{W})}{\partial \epsilon} = \left[ \left( 2\epsilon^{-3}f - \epsilon^{-2}\frac{2}{k}\epsilon^{\frac{2}{k}-1}(V^2 - \epsilon^2U^2)f' + 2\epsilon^{-2}\epsilon^{1+\frac{2}{k}}U^2f' \right) V + \frac{\hat{V}}{(m+\omega)^2}\frac{\epsilon}{\omega} \right], \quad (6.14)$$

$$\left( \frac{2}{k}\epsilon^{\frac{2}{k}-1}V^2 - \frac{2+2k}{k}\epsilon^{1+\frac{2}{k}}U^2 \right) U f' + \hat{U}\frac{\epsilon}{\omega}$$

with  $f, f'$  evaluated at  $\tau = \epsilon^{2/k}(V^2 - \epsilon^2U^2)$ . We recall that  $\tilde{W} = \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix}$ ,  $V = \hat{V} + \tilde{V}$ ,  $U = \hat{U} + \tilde{U}$ ; cf. (2.7), (2.14). By (6.1), taking into account the exponential decay of  $\hat{V}$  and  $\hat{U}$ , and also  $\|e^{\gamma(t)}\tilde{W}\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = O(\epsilon^{2\kappa})$  (cf. Theorem 2.1 (5)), one has:

$$\begin{aligned} \|e^{\gamma(t)}(f(\tau) - \frac{\tau}{k}f'(\tau))\|_X &= \|e^{\gamma(t)}O(|\tau|^K)\|_X = O(\epsilon^{2K/k}), \\ \|e^{\gamma(t)}\epsilon^{2/k}U^2f'(\tau)\|_X &\leq C\|e^{\gamma(t)}\epsilon^{2/k}V^2f'(\tau)\|_X \\ &\leq C\|e^{\gamma(t)}\tau f'(\tau)\|_X = \|O(|\tau|^k)\|_X = O(\epsilon^2), \end{aligned}$$

where  $\tau(t) = \epsilon^{2/k}(V(t)^2 - \epsilon^2U(t)^2)$ . Applying these estimates to terms in (6.14), one arrives at the second estimate stated in the lemma.  $\square$

Multiplying (6.12) by  $e^{\gamma(t)}$ , we have:

$$e^{\gamma(t)}\partial_\epsilon \tilde{W} = B^{-1} \circ e^{\gamma(t)} \circ A^{-1} \circ e^{-\gamma(t)} \circ (\partial_\epsilon A \circ e^{\gamma(t)} \circ \tilde{W} + e^{\gamma(t)}\partial_\epsilon G(\epsilon, \tilde{W})). \quad (6.15)$$

Above,  $e^{\pm\gamma(t)}$  are understood as the multiplication operators; we note that they commute with

$$\partial_\epsilon A(\epsilon) = \frac{\epsilon}{\omega} \begin{bmatrix} -\frac{1}{(m+\omega)^2} & 0 \\ 0 & -1 \end{bmatrix}.$$

The operator  $B(t, \epsilon)$  (cf. (6.13)) defines a mapping

$$B(t, \epsilon)^{-1} : X^1 \rightarrow X^1 \quad (6.16)$$

which is continuous since both  $\|B(t, \epsilon)^{-1}\|_{\text{End}(\mathbb{C}^2)}$  and  $\|\partial_t B(t, \epsilon)\|_{\text{End}(\mathbb{C}^2)}$  are bounded uniformly in  $t \in \mathbb{R}$  and  $\epsilon \in (0, \epsilon_2)$ , as long as  $\epsilon_2 > 0$  is sufficiently small; we took into account that  $\|\partial_{\tilde{W}} G(\epsilon, \tilde{W})\|_{\text{End}(\mathbb{C}^2)} = O(\epsilon^{2\kappa})$  by Lemma 6.2, while the derivatives  $\partial_t V(t, \epsilon)$  and  $\partial_t U(t, \epsilon)$  are bounded pointwise, uniformly in  $t \in \mathbb{R}$  and  $\epsilon \in (0, \epsilon_2)$ , due to Lemma 3.9, and hence so is  $\partial_t \tilde{W}(t, \epsilon)$ .

Since  $\|e^{\gamma\langle t \rangle} \tilde{W}\|_{X^1} = O(\epsilon^{2\kappa})$  (cf. Lemma 5.4) and the mapping  $e^{\gamma\langle t \rangle} \circ A(\epsilon)^{-1} \circ e^{-\gamma\langle t \rangle} : X \rightarrow X^1$  is continuous (just like the mapping  $e^{(1+2k)\gamma\langle t \rangle} \circ A(\epsilon)^{-1} \circ e^{-(1+2k)\gamma\langle t \rangle} : X \rightarrow X^1$  in (3.41)), while (6.16) is continuous in  $X^1$ , it follows that the  $X^1$ -norm of the right-hand side of (6.15) is bounded by

$$C \left( \epsilon \|e^{\gamma\langle t \rangle} \tilde{W}(t, \epsilon)\|_{X^1} + \|e^{\gamma\langle t \rangle} \partial_\epsilon G(\epsilon, \tilde{W}(t, \epsilon))\|_X \right) = O(\epsilon^{1+2\kappa}) + O(\epsilon^{2\kappa-1}) = O(\epsilon^{2\kappa-1}), \quad \epsilon \in (0, \epsilon_2);$$

we estimated the second term in the left-hand side with the aid of Lemma 6.4. Thus, the relation (6.15) gives

$$\partial_\epsilon \tilde{W} \in X_{e,o}^1, \quad \|e^{\gamma\langle t \rangle} \partial_\epsilon \tilde{W}\|_{X^1} = O(\epsilon^{2\kappa-1}), \quad \epsilon \in (0, \epsilon_2), \quad (6.17)$$

proving (2.27).

We can now estimate  $\|\partial_\omega \phi_\omega\|_{L^2}^2$ . We have:

$$\|\partial_\omega \phi_\omega\|_{L^2}^2 = \frac{\epsilon^2}{\omega^2} \left\| \frac{d}{d\epsilon} \phi_\omega \right\|_{L^2}^2 = \frac{\epsilon^2 \text{vol}(\mathbb{S}^{n-1})}{\omega^2} \int_0^\infty \left( (\partial_\epsilon(\epsilon^{\frac{1}{k}} V(\epsilon r, \epsilon)))^2 + (\partial_\epsilon(\epsilon^{1+\frac{1}{k}} U(\epsilon r, \epsilon)))^2 \right) r^{n-1} dr.$$

Let us estimate the above integral. Since  $\partial_\epsilon(\epsilon^{\frac{1}{k}} V(\epsilon r, \epsilon)) = \frac{1}{k} \epsilon^{\frac{1}{k}-1} V(\epsilon r, \epsilon) + \epsilon^{\frac{1}{k}} r \partial_t V(\epsilon r, \epsilon) + \epsilon^{\frac{1}{k}} \partial_\epsilon V(\epsilon r, \epsilon)$ , we have:

$$\begin{aligned} \int_0^\infty (\partial_\epsilon(\epsilon^{\frac{1}{k}} V(\epsilon r, \epsilon)))^2 r^{n-1} dr &= \epsilon^{-n} \int_0^\infty \left( \frac{\epsilon^{\frac{1}{k}-1}}{k} V(t, \epsilon) + \epsilon^{\frac{1}{k}-1} t \partial_t V(t, \epsilon) + \epsilon^{\frac{1}{k}} \partial_\epsilon V(t, \epsilon) \right)^2 t^{n-1} dt \\ &= \epsilon^{-n+\frac{2}{k}-2} \int_0^\infty \left( \frac{V(t, \epsilon)}{k} + t \partial_t V(t, \epsilon) + \epsilon \partial_\epsilon V(t, \epsilon) \right)^2 t^{n-1} dt = \epsilon^{-n+\frac{2}{k}-2} (C + O(\epsilon^{2\kappa})), \end{aligned}$$

with

$$C = \int_0^\infty \left( \frac{\hat{V}(t)}{k} + t \partial_t \hat{V}(t) \right)^2 t^{n-1} dt > 0.$$

We used Theorem 2.1 (5) for the  $L^2$ -norm of  $t \partial_t V(t, \epsilon)$  and (2.27) for the  $L^2$ -norm of  $\partial_\epsilon V(t, \epsilon) = \partial_\epsilon \tilde{V}(t, \epsilon)$ . We omit the computations for the part containing  $U$  since its contribution will be of the order  $O(\epsilon^2)$  smaller, which is dominated by the  $O(\epsilon^{2\kappa})$  error term. It follows that

$$\|\partial_\omega \phi_\omega\|_{L^2}^2 = \frac{\epsilon^2}{\omega^2} \|\partial_\epsilon \phi_\omega\|_{L^2}^2 = \epsilon^{-n+\frac{2}{k}} \frac{\text{vol}(\mathbb{S}^{n-1})}{\omega^2} (C + O(\epsilon^{2\kappa})),$$

proving (2.28).

This completes the proof of Theorem 2.2 (1).

## 7 Vakhitov–Kolokolov condition for the nonlinear Dirac equation

Finally, let us prove Theorem 2.2 (2). We start with the focusing nonlinear Schrödinger equation in  $n$  dimensions:

$$i\dot{\psi} = -\frac{1}{2m} \Delta \psi - |\psi|^{2k} \psi, \quad \psi(x, t) \in \mathbb{C}, \quad x \in \mathbb{R}^n. \quad (7.1)$$

Above,  $k \in (0, n/(n-2))$  (any  $k > 0$  if  $n \leq 2$ ). Given a positive solution  $u_k$  to the stationary Schrödinger equation

$$-\frac{1}{2m}u_k = -\frac{1}{2m}\Delta u_k - u_k^{1+2k}$$

(cf. (2.4)), one can use  $u_k$  to construct the solitary wave solutions to (7.1) for any  $\omega < 0$ :

$$\varphi_\omega(x) = (2m|\omega|)^{1/(2k)} u_k(\sqrt{2m|\omega|}x).$$

When  $k = 2/n$ , it follows that the  $L^2$ -norm of  $\varphi_\omega$  does not depend on  $\omega$ ;  $\frac{d}{d\omega}\|\varphi_\omega\|^2 = 0$ .

We are going to show that in the case of the nonlinear Dirac equation in  $(n+1)\text{D}$  with the “critical” value  $k = 2/n$  (and absent or sufficiently small higher order terms), the charge is no longer constant; instead,  $\partial_\omega Q(\phi_\omega) < 0$  for  $\omega \lesssim m$ . This reduces the degeneracy of the zero eigenvalue of the linearization at the corresponding solitary wave; see e.g. [BCS15].

**Lemma 7.1.** *Assume that  $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$  satisfies the assumption (6.1) with some  $K > k > 0$ . One has:*

$$\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle = \epsilon q_1 + \epsilon \left( \frac{1}{k} - \frac{n}{2} \right) q_2 + O(\epsilon^{\frac{2K}{k}-3} + \epsilon^{4\kappa-1}) + o(\epsilon), \quad \epsilon \in (0, \epsilon_2),$$

with

$$q_1 = \int_{\mathbb{R}^n} (4m \hat{V}^{2k} \hat{U}^2 + \hat{U}^2) dy > 0, \quad q_2 = \int_{\mathbb{R}^n} \left( \frac{\hat{V}^2}{4m^2} + 2m \hat{V}^{2k} \hat{U}^2 + \hat{U}^2 \right) dy > 0.$$

*Proof.* By (2.6),  $\mathfrak{l}_+(\frac{1}{k}\hat{V} + x \cdot \nabla \hat{V}) = -\frac{1}{m}\hat{V}$ ; hence,  $A(0) \begin{bmatrix} \frac{1}{k}\hat{V} + x \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + x \cdot \nabla \hat{V}) \end{bmatrix} = \frac{1}{m} \begin{bmatrix} \hat{V} \\ 0 \end{bmatrix}$ . Therefore,

$$\begin{aligned} \frac{\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle}{m} &= \left\langle \frac{1}{m} \begin{bmatrix} \hat{V} \\ 0 \end{bmatrix}, \partial_\epsilon \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\rangle = \left\langle A(0) \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, \partial_\epsilon \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, A(0)\partial_\epsilon \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, A(\epsilon)\partial_\epsilon \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix} \right\rangle + O(\epsilon^2)\|\partial_\epsilon \tilde{W}\|_{L^\infty}. \end{aligned}$$

We took into account that the operator  $A(\epsilon)$  defined in (3.18) is self-adjoint on  $X_{e,o}^1$  and that  $\|A(\epsilon) - A(0)\|_{L^\infty(\mathbb{R}, \text{End}(\mathbb{C}^2))} = O(\epsilon^2)$ . Taking the derivative of (3.19) with respect to  $\epsilon$ , we derive:

$$\begin{aligned} \frac{\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle}{m} &= \left\langle \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, \partial_{\tilde{W}} G \partial_\epsilon \tilde{W} + \partial_\epsilon G - \partial_\epsilon A(\epsilon) \tilde{W} \right\rangle + O(\epsilon^2)\|\partial_\epsilon \tilde{W}\|_{L^\infty} \\ &= \left\langle \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, \partial_\epsilon G \right\rangle + O(\epsilon^{4\kappa-1} + \epsilon^{2\kappa+1}). \end{aligned}$$

We used the estimates  $\|\partial_{\tilde{W}} G\|_{L^\infty(\mathbb{R}, \text{End}(\mathbb{C}^2))} = O(\epsilon^{2\kappa})$  (cf. Lemma 6.2),  $\|\tilde{W}\|_{L^\infty} = O(\epsilon^{2\kappa})$  (cf. Theorem 2.1 (5)), and  $\|\partial_\epsilon \tilde{W}\|_{L^\infty} = O(\epsilon^{2\kappa-1})$  (cf. Theorem 2.2 (I)).

Taking into account Lemma 6.4 to express  $\partial_\epsilon G(\epsilon, \tilde{W})$ , we continue:

$$\begin{aligned} \frac{\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle}{m} &= \epsilon \left\langle \begin{bmatrix} \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \\ -\frac{1}{2m}\partial_r(\frac{1}{k}\hat{V} + y \cdot \nabla \hat{V}) \end{bmatrix}, \begin{bmatrix} 2k\hat{U}^2\hat{V}^{2k-1} + \frac{\hat{V}}{4m^3} \\ 2\hat{U}\hat{V}^{2k} + \frac{\hat{U}}{m} \end{bmatrix} \right\rangle + O(\epsilon^{\frac{2K}{k}-3} + \epsilon^{4\kappa-1}) + o(\epsilon) \\ &= \epsilon \int_{\mathbb{R}^n} \left\{ \left( \frac{1}{k}\hat{V} + y \cdot \nabla \hat{V} \right) \left( 2k\hat{V}^{2k-1}\hat{U}^2 + \frac{\hat{V}}{4m^3} \right) + \left( \frac{1+k}{k}\hat{U} + y \cdot \nabla \hat{U} \right) \left( 2\hat{V}^{2k}\hat{U} + \frac{\hat{U}}{m} \right) \right\} dy \\ &\quad + O(\epsilon^{\frac{2K}{k}-3} + \epsilon^{4\kappa-1}) + o(\epsilon); \end{aligned}$$

we took into account that  $-\frac{1}{2m}\partial_r\left(\frac{1}{k}\hat{V} + y\cdot\nabla\hat{V}\right) = \frac{1}{k}\hat{U} + y\cdot\nabla\hat{U} + \hat{U}$ . The integral  $\int_{\mathbb{R}^n}\{\dots\}dy$  is evaluated by parts as follows:

$$\begin{aligned} & \int_{\mathbb{R}^n} \left\{ \frac{\hat{V}^2}{4m^3k} + 2\hat{V}^{2k}\hat{U}^2 + \frac{y\cdot\nabla\hat{V}^2}{8m^3} + y\cdot\nabla(\hat{V}^{2k}\hat{U}^2) + \frac{(1+k)\hat{U}^2}{mk} + \frac{2(1+k)\hat{V}^{2k}\hat{U}^2}{k} + \frac{y\cdot\nabla\hat{U}^2}{2m} \right\} dy \\ &= \int_{\mathbb{R}^n} \left\{ \frac{\hat{V}^2}{4m^3k} + 2\hat{V}^{2k}\hat{U}^2 - \frac{n\hat{V}^2}{8m^3} - n\hat{V}^{2k}\hat{U}^2 + \frac{(1+k)\hat{U}^2}{mk} + \frac{2(1+k)\hat{V}^{2k}\hat{U}^2}{k} - \frac{n\hat{U}^2}{2m} \right\} dy \\ &= \int_{\mathbb{R}^n} \left[ \frac{\hat{V}^2}{4m^3} \left( \frac{1}{k} - \frac{n}{2} \right) + 2 \left( 2 + \frac{1}{k} - \frac{n}{2} \right) \hat{V}^{2k}\hat{U}^2 + \left( 1 + \frac{1}{k} - \frac{n}{2} \right) \frac{\hat{U}^2}{m} \right] dy. \quad \square \end{aligned}$$

**Lemma 7.2.** *Let  $f \in C^1(\mathbb{R} \setminus \{0\})$  satisfy  $f(\tau) = |\tau|^k + O(|\tau|^K)$ ,  $\tau \in \mathbb{R}$ .*

1. *Assume that in the assumption (6.1) either  $k \in (0, 2/n)$ , or  $k = 2/n$ ,  $K > 4/n$ . Then there is  $\omega_* \in (\omega_2, m)$  such that  $\partial_\omega Q(\phi_\omega) < 0$  for  $\omega \in (\omega_*, m)$ .*
2. *If in the assumption (6.1) one has  $k \in (2/n, 2/(n-2))$  (any  $k > 2/n$  if  $n \leq 2$ ), then there is  $\omega_* \in (\omega_2, m)$  such that  $\partial_\omega Q(\phi_\omega) > 0$  for  $\omega \in (\omega_*, m)$ .*

Above,  $\omega_2 = \sqrt{m^2 - \epsilon_2^2}$ , with  $\epsilon_2 > 0$  from Theorem 2.2 (I).

*Proof.* We recall that  $v(x, \omega) = \epsilon^{\frac{1}{k}}(\hat{V}(\epsilon x) + \tilde{V}(\epsilon x, \epsilon))$ ,  $u(x, \omega) = \epsilon^{\frac{1}{k}+1}(\hat{U}(\epsilon x) + \tilde{U}(\epsilon x, \epsilon))$  (cf. Theorem 2.1);

$$Q(\phi_\omega) = \int_{\mathbb{R}^n} |\phi_\omega(x)|^2 dx = \epsilon^{\frac{2}{k}-n} \int_{\mathbb{R}^n} (V(|y|, \epsilon)^2 + \epsilon^2 U(|y|, \epsilon)^2) dy.$$

Let us evaluate the contribution to the derivative of  $Q(\phi_\omega)$  with respect to  $\epsilon$ :

$$\begin{aligned} \partial_\epsilon Q &= \left( \frac{2}{k} - n \right) \epsilon^{\frac{2}{k}-n-1} (\langle V, V \rangle + \epsilon^2 \langle U, U \rangle) + \epsilon^{\frac{2}{k}-n} \partial_\epsilon (\langle V, V \rangle + \epsilon^2 \langle U, U \rangle) \\ &= \left( \frac{2}{k} - n \right) \epsilon^{\frac{2}{k}-n-1} (\langle V, V \rangle + \epsilon^2 \langle U, U \rangle) + \epsilon^{\frac{2}{k}-n} \left( 2\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle + 2\epsilon \langle \hat{U}, \hat{U} \rangle + O(\epsilon^{4\kappa-1}) \right). \end{aligned}$$

The estimate on the error terms in the right-hand side, such as  $\langle \tilde{V}, \partial_\epsilon \tilde{V} \rangle = O(\epsilon^{4\kappa-1})$ , follows from (2.13), (2.14), and  $X^1$ -bounds on  $\tilde{W}$  and  $\partial_\epsilon \tilde{W}$  from Theorem 2.1 (5) and Theorem 2.2 (I), respectively. By Lemma 7.1, in the non-critical case, when  $k \neq 2/n$  and  $K > k$ , one has

$$\partial_\epsilon Q = \left( \frac{2}{k} - n \right) \epsilon^{\frac{2}{k}-n-1} \langle \hat{V}, \hat{V} \rangle + O(\epsilon^{\frac{2}{k}-n} \epsilon^{\frac{2K}{k}-3}) + o(\epsilon^{\frac{2}{k}-n-1}) = \left( \frac{2}{k} - n \right) \epsilon^{\frac{2}{k}-n-1} \langle \hat{V}, \hat{V} \rangle + o(\epsilon^{\frac{2}{k}-n-1});$$

hence, for  $\epsilon > 0$  sufficiently small, the sign of  $\partial_\epsilon Q$  is determined by the sign of  $\frac{2}{k} - n$ . Thus, if  $k \in (0, 2/n)$ , one has  $\partial_\omega Q = -\frac{\epsilon}{\omega} \partial_\epsilon Q < 0$  as long as  $\omega < m$  is sufficiently close to  $m$ . In the critical case  $k = 2/n$ , again by Lemma 7.1,

$$\begin{aligned} \partial_\epsilon Q(\omega) &= 2\langle \hat{V}, \partial_\epsilon \tilde{V} \rangle + 2\epsilon \langle \hat{U}, \hat{U} \rangle + O(\epsilon^{4\kappa-1}) \\ &= 2\epsilon \int_{\mathbb{R}^n} (4m\hat{V}^{2k}\hat{U}^2 + \hat{U}^2) dy + 2\epsilon \int_{\mathbb{R}^n} \hat{U}^2 dy + O(\epsilon^{\frac{2K}{k}-3} + \epsilon^{4\kappa-1} + \epsilon^{2\kappa+1}) + o(\epsilon). \end{aligned}$$

If  $K/k > 2$ ,  $\kappa = \min\left(1, \frac{K}{k} - 1\right) = 1$ , then, for  $\epsilon > 0$  sufficiently small, the above is dominated by the first term of order one in  $\epsilon$ , hence is strictly positive. Thus, in this case,  $\partial_\omega Q = -\frac{\epsilon}{\omega} \partial_\epsilon Q < 0$  as long as  $\omega < m$  is sufficiently close to  $m$ . This finishes the proof of Lemma 7.2.  $\square$

This concludes the proof of Theorem 2.2 (2).



## A Appendix: smoothness of NLS groundstates

We start with the properties of the profiles of solitary wave solutions to the nonlinear Schrödinger equation.

**Lemma A.1.** *Let  $n \geq 1$  and  $k > 0$ . If  $n \geq 3$ , additionally assume that  $k < \frac{2}{n-2}$ . Then there is a unique positive spherically symmetric monotonically decaying solution  $u_k \in H^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$  to the equation*

$$-\frac{u}{2m} = -\frac{\Delta u}{2m} - |u|^{2k}u, \quad u(x) \in \mathbb{R}, \quad x \in \mathbb{R}^n. \quad (\text{A.1})$$

For any  $\delta < 1$  there is  $C_\delta < \infty$  such that

$$|u_k(r)| + |\partial_r u_k(r)| \leq C_\delta e^{-\delta r}, \quad r \geq 0.$$

For any  $s < \frac{n}{2} + 2$  one has  $u_k \in H^s(\mathbb{R}^n)$ . As  $|x| \rightarrow \infty$ , the function  $u_k$  is strictly monotonically decreasing. There are  $0 < c_n < C_n < \infty$  such that

$$\frac{c_n}{r^{(n-1)/2}} e^{-r} \leq |u_k(r)| \leq \frac{C_n}{r^{(n-1)/2}} e^{-r}, \quad r \geq 0. \quad (\text{A.2})$$

If  $n \geq 3$  and  $k \geq \frac{2}{n-2}$ , then (A.1) has no  $H^1$  solutions.

Let us give an extension of Lemma A.1, deriving optimal regularity of the groundstates of the nonlinear Schrödinger equation in Sobolev spaces.

*Proof.* The absence of  $H^1$ -solutions for  $k \geq \frac{2}{n-2}$ ,  $n \geq 3$  is proved in [BL83a, Section 2.1] via Pohozaev's identities. The uniqueness of a symmetric solution  $u > 0$  is proved in [Kwo89, McL93]. The inclusion  $u_k \in H^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ , monotonicity, and the exponential decay of  $u_k$  follows from [BL83a] for  $n \geq 3$  and  $n = 1$ ; for  $n = 2$ , the inclusion  $u_k \in H^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$  is proved in [BGK83], and the exponential decay is proved following the lines of [BC16, Lemma 3.1].

The exponential decay of  $\partial_r u$  could be shown as follows. The groundstate profile  $u_k$ , considered as a function of  $r = |x|$ , satisfies the equation

$$-\partial_r^2 u_k - \frac{n-1}{r} \partial_r u_k - 2m u_k^{2k+1} + u_k = 0, \quad r > 0.$$

Multiplying this by  $r^{n-1}$ , one has:

$$-\partial_r (r^{n-1} \partial_r u_k) - 2m r^{n-1} u_k^{2k+1} + r^{n-1} u_k = 0. \quad (\text{A.3})$$

Integrating this relation from zero to some  $R > 0$  and taking into account the exponential decay of  $u_k$ , one concludes that there exists a finite limit  $c = \lim_{r \rightarrow \infty} r^{n-1} \partial_r u_k$ . This limit has to be equal to zero or else there is  $r_0 > 0$  such that  $|r^{n-1} \partial_r u_k| \geq c/2$  for  $r \geq r_0$ , hence  $\partial_r u_k \leq -c/(2r^{n-1})$ ,  $u_k(r) \geq c/(2(n-2)r^{n-2})$  for  $r \geq r_0$  for  $n \neq 2$  or  $u_k(r) \geq (c \ln r)/2$  for  $r \geq r_0$  for  $n = 2$  or  $\lim_{r \rightarrow \infty} \partial_r u_k = c > 0$  for  $n = 1$ ; so, for any  $n \geq 1$ , we arrive at a contradiction with the exponential decay of  $u_k$ . So,  $\lim_{r \rightarrow \infty} r^{n-1} \partial_r u_k = 0$ . Integrating (A.3) from some  $R > 0$  to infinity, one has:

$$R^{n-1} \partial_r u_k(R) = \int_R^\infty (2m u_k^{2k+1} - u_k) r^{n-1} dr.$$

Now the exponential decay of  $\partial_r u_k(r)$  follows from the exponential decay of  $u_k(r)$ .

The strict monotonicity of  $u_k$  is proved as follows. Assume that

$$\partial_r u_k = w_k, \quad \partial_r w_k = -\frac{n-1}{r} w_k - 2m |u_k|^{2k} u_k, \quad r > 0, \quad (\text{A.4})$$

and that  $u'_k(r_0) = 0$  (here  $u_k$  is considered as a function of  $r = |x|$ ) at some  $r_0 > 0$ . Since  $u_k(r)$  is monotonically decreasing,  $w_k = \partial_r u_k \in C^1(\mathbb{R}_+)$  satisfies  $w_k \leq 0$ . Once we know that  $w_k(r_0) = u'_k(r_0) = 0$ , we conclude that  $w_k$  has a local maximum at  $r_0$ , so that  $\partial_r w_k(r_0) = 0$ . Now from the second equation in (A.4) one would conclude that  $u_k(r_0) = 0$ , in contradiction to the strict positivity of the groundstate  $u_k$ .

The estimate (A.2) follows from Lemma 4.5.

Let us prove the improved Sobolev regularity

$$u_k \in H^s(\mathbb{R}^n), \quad \forall s < \frac{n}{2} + 2.$$

Considering  $u$  as a function of  $r \geq 0$ , we write (A.1) in the form

$$u'' = u - 2mu^{1+2k} - \frac{n-1}{r}u', \quad r > 0. \quad (\text{A.5})$$

Denote

$$f(r) = \frac{u'(r)}{u(r)};$$

note that  $f$  is non-positive since  $u$  is non-increasing.

**Lemma A.2.** *There is  $c_1 < \infty$  such that*

$$|f(r)| = \left| \frac{u'(r)}{u(r)} \right| \leq c_1 \frac{r}{\langle r \rangle}, \quad r > 0. \quad (\text{A.6})$$

*Proof.* Using (A.5), we arrive at

$$f'(r) = \frac{u''}{u} - f^2 = 1 - 2mu^{2k} - \frac{n-1}{r}f - f^2, \quad r > 0. \quad (\text{A.7})$$

We already mentioned that  $f(r) \leq 0$ ,  $r > 0$ . If  $f(r)$  were unbounded from below for  $r \geq 1$ , then it would blow up, going to  $-\infty$  at some  $r_0 < \infty$ . Indeed, fix  $a = \sup_{r>0} |1 - 2mu^{2k}| < \infty$ , and, assuming  $f \rightarrow -\infty$ , consider the smallest  $r_1 \geq 1$  such that  $-f(r) \geq 4 \max\{\frac{n-1}{r}, \sqrt{a}\}$  for  $r \geq r_1$ ; then  $|f(r)|$  grows faster than the solution to  $F' = F^2/2 - a/2$  with the same initial data  $F(r_1) = f(r_1)$ , while this solution blows up in the interval  $[r_1, r_1 - 4/f(r_1)]$ . Of course, the blow-up of  $f$  at some  $r < \infty$  would contradict  $u \in C^2$ . We conclude that  $|f|$  remains bounded as  $r \rightarrow +\infty$ . We also conclude from (A.5) and from the inclusion  $u \in C^2(\mathbb{R}^n)$  (considered as a function of  $x \in \mathbb{R}^n$ ) that  $u'/r$  remains bounded near  $r = 0$ ; due to  $u(0) > 0$ , the bound (A.6) follows.  $\square$

We claim that for  $j \geq 2$  there are  $C_j < \infty$  such that

$$|u^{(j)}(r)| \leq C_j \left( \frac{\langle r \rangle}{r} \right)^{j-2} u(r), \quad r > 0, \quad j \geq 1. \quad (\text{A.8})$$

The proof is by induction. For  $j = 2$ , the statement follows from (A.5) and Lemma A.2. Assume that (A.8) is proved for  $j \leq l$ , with some  $l \in \mathbb{N}$ . To get  $u^{(l+1)}$  out of (A.5), one takes the derivative of the expression for  $u^{(l)}$ ,

$$u^{(l+1)} = u^{(l-1)} - \left( 2mu^{1+2k} + (n-1)\frac{u'}{r} \right)^{(l-1)}. \quad (\text{A.9})$$

We notice that each of  $l-1$  derivatives of the expression in the brackets, when acting on  $u$ , contributes a factor of  $u'/u$  (which is uniformly bounded); or else it changes one of the factors  $u^{(i)}$  to  $u^{(i+1)}$  with  $i < l$  (worsening the bound by  $\langle r \rangle/r$  by the induction assumptions); or else it acts on  $1/r$ , contributing another  $1/r$ ; therefore, after each differentiation, the resulting estimate deteriorates by the factor  $C\langle r \rangle/r$ , with some  $C < \infty$ . This allows to bound (A.9) by  $(\langle r \rangle/r)^{l-1}$  (times a constant factor), concluding the induction argument.

The inequality (A.8) and the interpolation arguments show that  $u \in H^s(\mathbb{R}^n)$  as long as  $|x|^{-(s-2)}$  is  $L^2$  locally near the origin; this imposes the restriction  $s - 2 < n/2$ .  $\square$

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